Ustozim Egamqul Abduqodirov va ukam Zayniddin Laqaevlarning xotirasiga bag'ishlayman.

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IKKI ZARRACHALI HAMILTONIANLAR VA FRIEDRICHS MODELLARI

Mundarija

KIRISH

Ushbu qo'llamnmaning asosiy maqsadi o'quvchilarni panjaradagi ikki zarrachali Schroedinger operatorlari va Friedrichs modellari hamda ularning spektral xossalari bilan tanishtirishdir.

Birinchi bo'limning asosiy maqsadlaridan biri $d-$ o'lchamli panjarada ikki zarrachali Hamiltonianlar uchun uzluksiz holda uchramaydigan yangi phenomenlarni o'rganishdir(muhokama uchun, [8], [33], [40]–[47], [48] lar va umuman pangaradagi sistemalar uchun quyi(pastki) qo'zg'algan spektrlar o'rganilgan ushbu [27], [36], [49], [62] [27], [36], [49], [62] ishlarni tavsiya qilamiz).

Panjaradagi kvant zarrachalari kinematikasi hattoki ikki zarrachali bo'limda ham juda ajoyibdir. Masalan panjaradagi Laplasian va uning umumlashmalari burishga nisbatan invariant emas, shu sababli harakatning massalar markazi va ichki erkinliklar darajalariga bog'liq ikki qismga ajralmaydi. Xususan oddiy massa inersiyasini ham aniqlashning iloji yo'q.

Panjarada ikki zarrachali masala (effektiv) bir zarrachali masalaga Gelfand almashtirishi yordamida keltiriladi: asosiy Hilbert fazosi va undagi ikki zarrachali Hamiltonian diskret $\mathbb{Z}^3$ guruhning bu fazodagi tasviri bo'lgan siljitish operatorlari yordamida fon-Neyman to'g'ri integraliga yoyiladi.

Natijada masalaning sferik simmetriklik xususiyati yo'qoladi va $h(k), k \in \mathbb{T}^d$, operatorlar oilasi spektri *quasi-impuls* k ning o'zgarishiga nisbatan ta'sirchan bo'ladi.

Xuddi shunday local effective massa tenzori ham sistema quasi-impulsiga bog'liq va faqat yarim additivdir. Bu hodisa panjaradagi "exess mass " phenomenoni deb ataladi ([47] va [48] ga q.).

$N-$ zarrachali sistema asosiy holati effective massasi shu sistema zarrachalari massalari yig'indisidan katta (umuman olganda teng emas).

Uzluksiz ($\mathbb{R}^3$) Schroedinger operatori uchun manfiy diskret spektr (xos qiymatlar) ning uzluksiz spektrda yutulishi hodisasi (kritik potensial bo'lgan hol) kuzatiladi ([3], [43], [51], [59] larga q.). Bu phenomenon Schroedinger tenglamasi nol energiyali umumlashgan yechimlari (cheksizlikda nolga intiluvchi, ammo integrallanuvchi bo'lmagan) bilan juda bog'liqdir. Bunday yechimlar odatda nol energiyali resonance funktsiyalar deyiladi.

Kritik (nomusbat) Schroedinger operatorlari uchun manfiy xos qiymatlarning (juda kichik manfiy qo'zg'alishlarda) paydo bo'lishi alohida e'tiborga loyiqdir, chunki agar uch zarrachali sistemada bironta ham ikki zarrachali qism sistema manfiy energiyali asosiy holatga ega bo'lmasa va kamida ikkitasi no'l energiyali resonanslarga ega bo'lsa, bunday uch zarrachali sistema manfiy energiyali cheksiz ko'p asosiy holatlar (xos qiymatlar)ga ega ([35][22][31], [4], [40],[50],[55], [56], [58] va [60] ga q.). .

Panjaradagi Schroedinger operatorlari uchun uzluksiz spektr tubi pastida xos qiymat (asosiy holat) ga ega bo'lishining kutilmagan imkoniyati mavjud: buning uchun potensialni umuman qo'zg'atish shart emas, faqatgina kinetik energiyani, ya'ni sistema " *quasi-impulsi*"ni ozgina qo'zg'atish kifoya.

Ikkinchi bo'limda $\mathbb{Z}^3$ panjarada qo'zg'alishi bir o'lchamli ikki zarrachali sistemaga mos Fridrixs modellari oilasini qaraymiz. Ishning asosiy maqsadi Fridriechs modellari oilasining spektral xossalarini mukammal matematik o'rganish va ushbu oilaga mos Fredgolm determinanti uchun bo'saga energiyali yoyilmalarni olishdan iborat.

Muhokama uchun [3, 5, 6, 8, 33, 40, 55, 56, 60] larga va panjaradagi kvant zarrachali sistema spektri qo'zg'alishini o'rganish uchun [36, 49, 62] larga qarang). Modellarning bu turi kvant mexanikasi [30, 34], qattiq jismlar fizikasi [52, 47, 48, 27] va panjaradagi maydonlar nazariyasi [46, 44, 45] da uchraydi.

Panjaradagi Shroedinger operatorlariga o'xshash, ammo uzluksiz ikki zarrachali Shroedinger operatorlariga zid ravishda Fridrixs modellari oilasi $h_\mu(p)$, $p \in (-\pi, \pi]^3$, $\mu > 0$ kvaziimpuls $p \in (-\pi, \pi]^3$ ga bog'liq va shuning uchun panjaradagi ikki zarrachali Schrödinger operatorlariga o'xshash spektral xossalarga ega. Asl Fridrixs modeli va uning umumlashmalarining spektral xossalari va rezolventalari [34, 30, 37, 61] ishlarda o'rganilgan va muhim spektr tubidan quyida yotgan xos qiymatlarining chekliligi isbotlangan.

[44, 45] ishlarda Fridrixs modellarining maxsus oilasi qaralgan va sistema to'la kvazi-impulsi $p, p \in (-\pi, \pi]^d, d = 1, 2$ ning bazi xususiy qiymatlari atrofida yotuvchi qiymatlar uchun xos qiymatlarning paydo bo'lishi ko'rsatilgan. Ushbu [6] da ikki zarrachali Shroedinger operatorlarining katta bir sinfi uchun kvaziimpuls ($0 \neq p \in \mathbb{T}^3$) ning barcha nolmas qiymatlarida ($h_\mu(0)$ operator yo nol energiyali rezonans yoki nol xos qiymatga ega bo'lsa) $h_\mu(p), p \in (-\pi, \pi]^3$ operatorning xos qiymatlari mavjudligi isbotlangan.

Mazkur bo'limda ikkita asosiy natija isbotlangan.

Birinchi natija $h_\mu(p), p \in (-\pi, \pi]^3$ operator kvaziimpuls $0 \neq p \in \mathbb{T}^3$ ning barcha no'lmas qiymatlarida yagona musbat $e_\mu(p)$ xos qiymatga ega ekanli va uning quyi va yuqori chegaralari mavjudligi hamda $e_\mu(p)$ xos qiymatning $\mu > 0$ ga monoton bog'liqligidan iborat.

Ikkinchi natija Fredholm determinanti yoki Birman- Shwinger operatori uchun koordinatalar boshining kichik $\delta-$ atrofida $p \in (-\pi, \pi]^3$ kvaziimpulsning darajalari boyicha yoyilmasini ifodalaydi. Xususan bu yoyilmalar Fredholm determinanti yoki Birman -Schwinger operatori uchun $w = (m - z)^{\frac{1}{2}} \geq 0$, $z \leq m$ o'zgaruvchining funktsiyasi sifatida $h_\mu(p)$ operatorning muhim spektri tubigacha differensiallanuvchan davomga ega ekanligini isbotlaydi (Teorema 14.15).

Shuni qayd qilamizki, agar u va v funktsiyalar mos ravishda $(\mathbb{T}^3)^2$ va $\mathbb{T}^3$, da analitik bo'lsa, u holda Fredholm determinani va Birman-Schwinger operatori uchun aniq yoyilmalarni olish mumkin.

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1. PANJARADAGI IKKI ZARRACHALI HAMILTONIANLAR UCHUN BO'SAG'A HODISASI

Kalit so'zlar: Panjara, Tor, Spektr,Diskret va muhim spektr,Musbat operator,Musbat operatorning kvadrat ildizi,Diskret Schrödinger operatori, ikki zarrachali kvant sistema, Hamiltonian,kontakt potentsial,Qisqa ta'sirli potentsial,Weyl teoremasi, Shartli manfiy aniqlangan funktsiyalar, dispersion munosabatlar, virtual sath, xos qiymat, panjara,Birman-Schwinger prinsipi,Hilbert-Schmidt sinfi,Fon-Neyman to'g'ri integrali,rezonans

2. BIR ZARRACHALI HAMILTONIAN

2.1. **Dispersion munosabatlar.** Faraz qilaylik $\mathbb{Z}^d-$ d-o'lchamli panjara va $\ell^2(\mathbb{Z}^d)$ kvadrati bilan jamlanuvchi funktsiyalarning Hilbert fazosi bolsin.

d-o'lchamli panjara $\mathbb{Z}^d$, $d \geq 1$ da bitta kvant zarrachaning $\hat{h}^0$ Hamiltoniani odatda $\ell^2(\mathbb{Z}^d)$ Hilbert fazosida (chegaralangan) o'z-o'ziga qo'shma ko'p o'lchamli Toeplitz-tipidagi operator yordamida beriladi:

$$(\hat{h}^0\hat{\psi})(x) = \sum_{s\in\mathbb{Z}^d} \hat{\varepsilon}(s)\hat{\psi}(x+s), \quad \hat{\psi} \in \ell^2(\mathbb{Z}^d). \tag{2.1}$$

Bunda $\sum_{s\in\mathbb{Z}^d} \hat{\varepsilon}(s)$ qator absolyut yaqinlashuvchi deb faraz qilinadi, ya'ni

$$\{\hat{\varepsilon}(s)\}_{s\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d).$$

Ushbu $\hat{\varepsilon}(\cdot)$ funktsiya "o'z-o'ziga qo'shma"lik sharti

$$\hat{\varepsilon}(s) = \overline{\hat{\varepsilon}(-s)}, \quad s \in \mathbb{Z}^d.$$

ni ham qanoatlaniradi.

Fizika adabiyotlarida $\hat{h}^0$ Toeplitz operatori simvoli $\varepsilon(\cdot)$ $d-$ o'lchamli to'r $\mathbb{T}^d$ da aniqlangan haqiqiy qiymatli funktsiya, ya'ni ushbu

$$\varepsilon(p) = \sum_{s\in\mathbb{Z}^d} \hat{\varepsilon}(s)e^{\mathrm{i}(p,s)}, \quad p \in \mathbb{T}^d, \tag{2.2}$$

Fourier qatori orqali beriladi va erkin zarracha *normal mode* ining dispersion *munosabatlari deyiladi.*

Eslatma 2.1. *Mazkur ish davomida $\mathbb{T}^d$ tor $\mathbb{R}^d$ Euiclid fazosidagi $(2\pi\mathbb{Z})^d$ modul bo'yicha kiritilgan qo'shish va songa ko'paytirish amallariga nisbatan abel guruhi sifatida qaraladi.*

Ta'kidlash lozimki ((2.2)dagi $\varepsilon(p)$ funktsiyaning Fourier koeffisentlari $\hat{\varepsilon}(s)$ koeffisentlaridan $(2\pi)^{\frac{d}{2}}$ ko'paytuvchiga farq qiladi.

Bir zarrachali erkin Hamiltonianning

$$\hat{h}^0 = \varepsilon(-\mathrm{i}\nabla),$$

ko'rinishda berilishi talab qilinadi, bunda ∇ *cheksiz almashtirishlarning tug'dirvchisi.*

Faraz qilaylik (F.q.) $\hat{v} = \{\hat{v}(s)\}_{s\in\mathbb{Z}^d}$ chegaralangan haqiqiy sonlar ketma-ketligi, ya'ni

$$\hat{v} \in \ell^\infty(\mathbb{Z}^d),$$

bolsin. U holda kvant zarrachaning $\hat{v}$ potensial maydondagi harakatini ifodalovchi bir zarrachali Hamiltonian

$$\hat{h} = \hat{h}^0 + \hat{v},$$

Hilbert fazosi $\ell^2(\mathbb{Z}^d)$ da chegaralangan o'z-o'ziga qo'shma operatordir.

F.q. $\mathcal{F} : L^2(\mathbb{T}^d) \longrightarrow \ell^2(\mathbb{Z}^d)$, standart Fourer almashtiri bo'lsin.

Impuls ko'rinishda h Hamiltonian

$$h = \mathcal{F}^{-1}\hat{h}\mathcal{F},$$

kabi tasvirlanadi.

2.2. **Musbatlikni saqlovchi yarim-guruhni tug'diruvchi (paydo qiluvchi) Hamiltonianlar.** Bizni bir zarrachali sistemaning quyidagi muhim sinfi ko'proq qiziqtiradi.

F.q haqiqiy qiymatli uzluksiz $\varepsilon(\cdot)$ dispersion munosabat shartli manfiy aniqlangan bo'lsin, xususan u

(i) juft
(ii) $p = 0$ da minimumga ega.

Ta'rif 2.2. *F.q. kompleks qiymatli* $\varepsilon : \mathbb{T}^d \longrightarrow \mathbb{C}$ *funktsiya berilgan bo'lsin. Agar ixtiyoriy* $n \in \mathbb{N}$, *barcha* $p_1, p_2, .., p_n \in \mathbb{T}^d$ *lar va*

$$\sum_{i=1}^{n} z_i = 0$$

tenglikni qanoatlantiruvchi barcha

$$\mathbf{z} = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$$

lar uchun

$$\sum_{i,j=1}^{n} \varepsilon(p_i - p_j) z_i \bar{z}_j \leq 0 \tag{2.3}$$

tengsizlik va $\varepsilon(p) = \overline{\varepsilon(-p)}$ *tenglik bajarilsa,* $\varepsilon : \mathbb{T}^d \longrightarrow \mathbb{C}$ *funktsiya shartli manfiy aniqlangan deyiladi.*

Ma'lumki, bu holda $\varepsilon(\cdot)$ dispersion munosabat

$$\varepsilon(p) = \varepsilon(0) + \sum_{s\in\mathbb{Z}^d\setminus\{0\}} (e^{\mathrm{i}(p,s)} - 1)\hat{\varepsilon}(s), \quad p \in \mathbb{T}^d,$$

yoyilmaga ega (Levy-Kinchin).

Bu yoyilma o'rinli bo'lishi $\hat{\varepsilon}(s)$, $s \neq 0$ Fourier koeffisentlarinig nomusbatligi, ya'ni

$$\hat{\varepsilon}(s) \leq 0, \quad s \neq 0,$$

va

$$\sum_{s\in\mathbb{Z}^d\setminus\{0\}} \hat{\varepsilon}(s)$$

qatorning absolyut yaqinlashuvchi bo'lish talablariga ekvivalent.

O'z navbatida bu $\ell^2(\mathbb{Z}^d)$ Hilbert fazosidagi

$$\hat{h} = \hat{h}^0 + \hat{v}$$

Hamiltonianning musbatlikni saqlovchi $e^{-t\hat{h}}$, $t > 0$, yarim guruhni paydo qilish (tug'dirish) shartiga ekvivalent.

Ta'rif 2.3. *Biz musbatlikni saqlovchi yarim guruhni paydo qiluvchi (tug'diruvchi)* $\hat{h}^0 = \varepsilon(-\mathrm{i}\nabla)$ *erkin Hamiltonianlarni panjaradagi Laplas operatorlari deb ataymiz.*

Quyidagi misol standart diskret Laplas operatori yuqorida keltirilgan ta'rifga ko'ra panjaradagi Laplas operatori bo'lishini ko'rsatadi.

Misol 2.4. *Ushbu bir zarrachali erkin Hamiltonian*

$$(\hat{h}^0\hat{\psi})(x) = (-\Delta\hat{\psi})(x) = \sum_{|s|=1} [\hat{\psi}(x) - \hat{\psi}(x+s)], \quad x \in \mathbb{Z}^d, \quad \hat{\psi} \in \ell^2(\mathbb{Z}^d),$$

uchun (2.1) *dagi* $\hat{\varepsilon}(s)$, $s \in \mathbb{Z}^d$, *Fourier koeffisientlari*

$$\hat{\varepsilon}(s) = \begin{cases} 2d, & s = 0 \\ -1, & |s| = 1 \\ 0, & \textit{aks holda}. \end{cases}$$

ko'rinishda bo'ladi.

Demak mos

$$\varepsilon(p) = 2\sum_{i=1}^{d} (1 - \cos p_i), \quad p = (p_1, p_2, \dots, p_d) \in \mathbb{T}^d, \tag{2.4}$$

dispersion munosabat yuqorida keltirilgan ekvivalentlik shartlariga ko'ra shartli manfiy aniqlangan funktsiya bo'ladi. (Buni ta'rifga ko'ra tekshirish ham foydali).

Quyida biz asosiy natijalarni (5.3 va 14.15 teoremalarni) isbotlashda muhim rol o'ynaydigan tengsizlikni isbotlaymiz:

Lemma 2.5. *Faraz qilaylik, $\varepsilon(\cdot)$ funktsiya quyidagi ikkita shartni qanoatlantirsin:*
1) $\varepsilon(\cdot)$-d o'lchamli tor $\mathbb{T}^d$ dagi haqiqiy qiymatli uzluksiz, sharti manfiy aniqlangan;
2) $\varepsilon(0)$ soni $\varepsilon(\cdot)$ funktsiyaning yagona minimumi bo'lsin.
U holda barcha $q \in \mathbb{T}^d \setminus \{0\}$ va deyarli barcha $p \in \mathbb{T}^d$ lar uchun

$$\varepsilon(p) + \varepsilon(q) > \frac{\varepsilon(p+q) + \varepsilon(p-q)}{2} + \varepsilon(0) \tag{2.5}$$

tengsizlik o'rinli.

Proof. $q \in \mathbb{T}^d$, $q \neq 0$ ni mahkamlaymiz. U holda shunday $s_0 \in \mathbb{Z}^d \setminus \{0\}$ mavjudki, $\hat{\varepsilon}(s_0) < 0$ va $\cos(q, s_0) \neq 1$ munosabatlar bajariladi (aks holda $\varepsilon(q) = \sum_{s\in\mathbb{Z}^d} \hat{\varepsilon}(s) = \varepsilon(0)$, bu esa 2) shartga zid).

Demak $\hat{\varepsilon}(s_0) < 0$ va $\cos(q, s_0) \neq 1$ bo'lgani uchun

$$\begin{aligned} F(p,q) &\equiv \varepsilon(p) + \varepsilon(q) - \frac{\varepsilon(p+q) + \varepsilon(p-q)}{2} - \varepsilon(0) \\ &= \sum_{s\in\mathbb{Z}^d} \hat{\varepsilon}(s)\left[\cos(p,s) + \cos(q,s) - \frac{\cos(p+q,s) + \cos(p-q,s)}{2} - 1\right] \\ &\geq 2\hat{\varepsilon}(s_0)\left[\cos(p,s_0) + \cos(q,s_0) - \frac{\cos(p+q,s_0) + \cos(p-q,s_0)}{2} - 1\right] \\ &= 2\hat{\varepsilon}(s_0)\Big[(\cos(p,s_0) - 1)(1 - \cos(q,s_0))\Big] > 0, \quad (p,s_0) \neq 2n\pi, \quad n \in \mathbb{Z}, \end{aligned} \tag{2.6}$$

Bu tengsizlik isbotni tugallaydi. □

3. VIRTUAL SATH VA BO'SAG'A HOS QIYMATLARI

Uch va to'rt o'lchamli ($d = 3, 4$) panjaralarda h energiya operatori uchun *virtual sath* (*bo'sag'a rezonansi*) tushunchasini kiritish uchun biz quyida $\varepsilon(p)$ dispersion munosabatning silliqligi va $\hat{v}$ o'zaro ta'sir Fourier almashtirishi

$$v(p) = (2\pi)^{-\frac{d}{2}} \sum_{s\in\mathbb{Z}^d} \hat{v}(s) e^{\mathrm{i}(p,s)}, \quad d \geq 3, \tag{3.1}$$

ning uzluksizligini ta'minlaydigan (kafolatlaydigan) texnik shart (farazlar) ni qabul qilamiz.

Talab 3.1. *1) $\varepsilon(p)$ dispersion munosabat $\mathbb{T}^d$ da uzluksiz (davriy) haqiqiy qiymatli va koordinatalar boshida yagona (aynimagan) minimumga ega bo'lgan funktsiya bo'lsin va*

$$\liminf_{|p|\to 0} \frac{\varepsilon(p) - \varepsilon(0)}{|p|^2} > 0$$

tengsizlik bajarilsin.
2) $v(p)$ funktsiya $\mathbb{T}^d$ torda uzluksiz va

$$v(p) = \overline{v(-p)}, \quad p \in \mathbb{T}^d$$

tenglikni qanoatlantiruvchi funktsiya bo'lsin.

3.1 talabga ko'ra $v(p)$ funktsiyaning Fourier koeffisientlarining $\{\hat{v}(s)\}_{s\in\mathbb{Z}^d}$ ketma-ketligi $\ell^2(\mathbb{Z}^d)$ ning elementi bo'lishini eslatamiz.

U holda (3.1) tenglikni $\ell^2(\mathbb{Z}^d)$ Hilbert fazosida uzluksiz $v(p)$ namoyondaga ega bo'lgan $(2\pi)^{-\frac{d}{2}} \sum_{s\in\mathbb{Z}^d} \hat{v}(s) e^{\mathrm{i}(p,s)}$ funktsiya sifatida tushunamiz.

Uzluksiz (davriy) funktsiyalarning $C(\mathbb{T}^d)$, $d \geq 3$ Banach fazosida yadro funktsiyasi

$$G(p,q;\lambda) = (2\pi)^{-\frac{d}{2}} v(p-q)(\varepsilon(q) - \lambda)^{-1}, \quad p,q \in \mathbb{T}^d. \tag{3.2}$$

bo'lgan $G(\lambda)$(Birman-Schwinger) integral operatorni qaraymiz.

Lemma 3.2. *$d \geq 3$ bo'lsin. Faraz qilaylik, 3.1 talab o'rinli bo'lsin. U holda $\lambda \leq \varepsilon(0)$ uchun $C(\mathbb{T}^d)$ da* (3.2) *kabi aniqlangan $G(\lambda)$ operator kompactdir.*

Proof. Berilgan $f \in L^1(\mathbb{T}^d)$ funktsiya uchun, quyidagi

$$|g(p)| \leq (2\pi)^{-\frac{d}{2}} \sup_{p,q\in\mathbb{T}^d} |v(p-q)| \|f\|_{L^1(\mathbb{T}^d)} \tag{3.3}$$

va

$$|g(p+\ell)-g(p)| = \left|(2\pi)^{-\frac{d}{2}}\int\limits_{\mathbb{T}^d}(v(p+\ell-q)-v(p-q))f(q)dq\right| \tag{3.4}$$

$$\leq (2\pi)^{-\frac{d}{2}}\sup_{t\in\mathbb{T}^d}|v(t+\ell)-v(t)|\|f\|_{L^1(\mathbb{T}^d)}.$$

baholashlar o'rinli bo'ladigan g funktsiyani

$$g(p)=(2\pi)^{-\frac{d}{2}}\int\limits_{\mathbb{T}^d}v(p-q)f(q)dq$$

kabi kiritamiz. O'lcham $d\geq 3$ bo'lganda $(\varepsilon(\cdot)-\lambda)^{-1},\lambda\leq\varepsilon(0)$ funktsiya integrallanuvchi bo'lganligi uchun, $C(\mathbb{T}^d)$ dan $L^1(\mathbb{T}^d)$ ga akslantiruvchi $(\varepsilon(\cdot)-\lambda)^{-1}$ funktsiyaga ko'paytirish operatori uzluksiz. Demak, (3.3) va (3.4) shartlardan birlik sharning $C(\mathbb{T}^d)$ G akslantirishdagi aksi (obrazi) to'la chegaralangan va tekis uzluksiz funktsiyalardan iborat ekanligi kelib chiqadi:yani v uzluksiz va demak,

$$\lim_{|\ell|\to 0}\sup_{t\in\mathbb{T}^d}|v(t+\ell)-v(t)|\|f\|_{L^1(\mathbb{T}^d)}=0.$$

Arzela-Ascoli teoremasini qo'llash Teorema isbotini yakunlaydi. □

Eslatma 3.3. *$d\geq 3$ bo'lsin. Ravshanki, (taqqoslang* [61]*), h operator $\lambda\leq\varepsilon(0)$ xos qiymatga ega, ya'ni, $\mathrm{Ker}(h-\lambda I)\neq 0$ bo'lishi uchun $G(\lambda)$ kompakt operator $C(\mathbb{T}^d)$ da -1 xos qiymatga ega va shunday $\psi\in\mathrm{Ker}(G(\lambda)+I)$ funktsiya topilib, deyarli hamma $p\in\mathbb{T}^d$ larda*

$$f(p)=\frac{\psi(p)}{\varepsilon(p)-\lambda} \tag{3.5}$$

ko'rinishda berilgan $f, f\in\mathrm{Ker}(h-\lambda I)$ funktsiya $L^2(\mathbb{T}^d)$ ga qarashli bo'lishi zarur va etarli. Bundan tashqari $\lambda<\varepsilon(0)$, bo'lsa u holda

$$\dim\mathrm{Ker}(h-\lambda I)=\dim\mathrm{Ker}(G(\lambda)+I) \tag{3.6}$$

va

$$\mathrm{Ker}(h-\lambda I)=\{f\,|\,f(\cdot)=\frac{\psi(\cdot)}{\varepsilon(\cdot)-\lambda}\,,\,\psi\in\mathrm{Ker}(G(\lambda)+I)\}.$$

Agar (3.6) *tenglik $d=3$ yoki $d=4$ bo'lganda bajarilsa, $\lambda=\varepsilon(0)$ bo'sag'a xos qiymat bo'ladi ($d\geq 5$ o'lchovlarda* (12.1) *kabi berilgan $f(p)$ funktsiya doimo $L^2(\mathbb{T}^d)$ ga qarashli bo'ladi. Quyidagi 3.6 lemma bilan taqqoslang). $d=3$ yoki $d=4$ o'lchovlarda* (3.6) *tenglik*

$$\dim\mathrm{Ker}(h-\varepsilon(0)I)\leq\dim\mathrm{Ker}(G(\varepsilon(0))+I).$$

tengsizlik bilan almashadi.

Bo'sag'a effektini, yani $\lambda=\varepsilon(0)$ holni muhokama qilish uchun biz quyida o'zaro bog'liqmas besh holni ajratamiz ([5],[26], [29] larga ham qarang):

I hol : -1 soni $G(\varepsilon(0))$ operatorning xos qiymati emas, yani

$$0=\dim\mathrm{Ker}(h-\lambda I)=\dim\mathrm{Ker}(G(\lambda)+I).$$

II hol: -1 soni $G(\varepsilon(0))$ operatorning oddiy xos qiymati va unga mos xos funktsiya ψ

$$\frac{\psi(\cdot)}{\varepsilon(\cdot)-\varepsilon(0)}\notin L^2(\mathbb{T}^d)$$

shartni qanoatlantiradi, ya'ni

$$0=\dim\mathrm{Ker}(h-\lambda I)\quad \text{va}\quad \dim\mathrm{Ker}(G(\lambda)+I)=1.$$

III hol : -1 soni $G(\varepsilon(0))$ operatorning xos qiymati va unga mos ψ xos funktsiyalarning har biri

$$\frac{\psi(\cdot)}{\varepsilon(\cdot)-\varepsilon(0)}\in L^2(\mathbb{T}^d)$$

shartlarni qanotlantiradi, ya'ni

$$1\leq\dim\mathrm{Ker}(h-\lambda I)=\dim\mathrm{Ker}(G(\lambda)+I).$$

IV hol : -1 soni $G(\varepsilon(0))$ ning karrali xos qiymati va unga mos ψ xos funktsiyalardan kamida bittasi

$$\frac{\psi(\cdot)}{\varepsilon(\cdot)-\varepsilon(0)}\notin L^2(\mathbb{T}^d)$$

shartni qanoatlantiradi, ya'ni

$$2 \leq \dim \mathrm{Ker}(G(\lambda)+I) \geq \dim \mathrm{Ker}(h-\lambda I)+1.$$

V hol : -1 soni $G(\varepsilon(0))$ ning karrali xos qiymati va

$$\dim \mathrm{Ker}(h-\lambda I)+2 \leq \dim \mathrm{Ker}(G(\lambda)+I).$$

Yuqorida berilgan klasifikatsiya (sinflash)yordamida $d = 3$ va $d = 4$ o'lchamlar uchun virtual sathning quyidagi ta'rifini keltiramiz. ($d \geq 5$ o'lchamlarda II, IV va V hollar ro'y bermaydi (Eslatma 12.2 ga qarang)).

Ta'rif 3.4. *$d = 3, 4$ bo'lsin. II, IV va V hollarda h operator bo'sag'ada virtual sathga ega deyiladi.*

Eslatma 3.5. *Bizning virtual sathga bergan ta'rifimiz uzluksiz bo'lgan holdagi ta'rif bilan ustma-ust tushadi. Masalan (*[3], [55], [58], [**?**], [61] *larga va u yerdagi ko'rsatmalarga qarang). $d = 1$ va $d = 2$ o'lchamlarda ham virtual sath tushunchasini kiritish mumkin. Ammo impuls ko'rinishda Birman-Schwinger yadrosi qo'shimcha bo'sag'a maxsusliklariga ega bo'ladi.Shu sababli bizning bu ishdagi yondashuvimiz quyi ($d = 1, 2$ olchamlarga to'g'ridan to'g'ri tadbiq etilmaydi).*

Lemma 3.6. *F.q. $d \geq 3$, $\psi \in \mathrm{Ker}(G(\varepsilon(0))+I)$ bo'lsin va 3.1 talab bajarilsin. U holda*

$$f(p) = \frac{\psi(p)}{\varepsilon(p)-\varepsilon(0)}, \quad p \in \mathbb{T}^d$$

funktsiya $L_w^{\frac{d}{2}}(\mathbb{T}^d)$ (kuchsiz) fazoga tegishli bo'ladi.

Proof. Agar

$$\sup_{t>0} t^q \mathrm{mes}\{p \mid \ |f(p)| > t\} < \infty$$

bo'lsa, f funktsiya L^q kuchsiz sinfga qarashli bo'lishini eslatib o'tamiz. 3.1 talabga ko'ra shunday musbat $C \equiv const$ mavjudki

$$\varepsilon(p)-\varepsilon(0) \geq C|p|^2, \quad p \in \mathbb{T}^d$$

bajariladi.

U holda $\psi(\cdot) \in C(\mathbb{T}^3)$ bo'lganligi uchun $t \to \infty$ da

$$\mathrm{mes}\{p \in \mathbb{T}^d \mid \ |f(p)| > t\} = \mathrm{mes}\{p \in \mathbb{T}^d \mid \ \frac{|\psi(p)|}{\varepsilon(p)-\varepsilon(0)} > t\}$$

$$\leq \mathrm{mes}\{p \in \mathbb{T}^d \mid \ \|\psi\|_{C(\mathbb{T}^d)} > C|p|^2 t\} = \mathcal{O}(t^{-\frac{d}{2}}) \quad ,$$

munosabatlar o'rinli. Bu esa isbotni tugallaydi. □

Natija 3.7. *$d = 3, 4$ bo'lsin. F.q. h Hamiltonian virtual sathga ega bo'lsin va unga mos ψ sath funktsiya uchun $\psi \in \mathrm{Ker}(G(\varepsilon(0))+I)$ va $\frac{\psi(\cdot)}{\varepsilon(\cdot)-\varepsilon(0)} \notin L^2(\mathbb{T}^d)$ shartlar bajarilsin.*

U holda (3.1 shartga ko'ra)

$$f(p) = \frac{\psi(p)}{\varepsilon(p)-\varepsilon(0)}, \quad p \in \mathbb{T}^d, \quad d = 3, 4, \tag{3.7}$$

funktsiya $L_w^{r(d)}(\mathbb{T}^d)$ ga qarashli bo'ladi,

bunda

$$r(d) = \begin{cases} \frac{3}{2}, & d = 3, \\ 2, & d = 4. \end{cases}$$

Xususan (3.7) *ko'rinishida berilgan f funktsiya h operatorning $L^1(\mathbb{T}^d)$ Banach fazosida $\varepsilon(0)$ xos qiymatiga mos xos funktsiyasi bo'ladi, ya'ni*

$$hf = \varepsilon(0)f$$

va demak deyarli barcha $p \in \mathbb{T}^d$, $\quad d = 3, 4$, larda

$$\varepsilon(p)f(p) + (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(p-q)f(q)dq = \varepsilon(0)f(p),$$

tenglama yechimga ega.

Eslatma 3.8. *Sodda hisoblashlar shuni ko'rsatadiki, f integrallanuvchi funktsiyaning $\hat{f}(s)$, $s \in \mathbb{Z}^d$, $d = 3,4$ Fourier koeffitsientlari bir jinsli*

$$\sum_{s\in\mathbb{Z}^d} \hat{\varepsilon}(s)\hat{f}(x+s) + (\hat{v}(x) - \varepsilon(0))\hat{f}(x) = 0, \quad x \in \mathbb{Z}^d,$$

cheksiz tenglamalar sistemasining yechimi bo'ladi va shu sababli (koordinata tasvirda)

$$\hat{h}\hat{f} = \varepsilon(0)\hat{f}$$

tenglama $\hat{f}$ yechimga ega ,biroq bu yechim $\ell^2(\mathbb{Z}^d)$ ga tegishli bo'lmaydi, ammo cheksizlikda

$$\lim_{|s|\to\infty} \hat{f}(s) = 0$$

shartni qanoatlantiradi (Riemann-Lebesgue teoremasiga ko'ra).

Eslatma 3.9. *Agar $\varepsilon(p)$ dispersion munosabatning juft funktsiya bo'lishi, ya'ni $\varepsilon(p) = \varepsilon(-p)$ ma'lum bo'lsa yoki Fourier koeffitsientlari juft bo'lsa, ya'ni*

$$\hat{\varepsilon}(s) = \hat{\varepsilon}(-s) \in \mathbb{R}, \quad s \in \mathbb{Z}^d$$

shartni qanoatlantirsa, u holda $G(p,q;\lambda)$ Birman-Schwinger yadrosi

$$G(p,q;\lambda) = \overline{G(-p,-q;\lambda)}$$

kabi qo'shimcha xossaga ega bo'ladi.

Shuning uchun agar $\psi \in \mathrm{Ker}(G(\lambda) + I)$, $\lambda \le \varepsilon(0)$ bo'lsa, $\varphi(p) = \overline{\psi(-p)}$ funktsiya ham shu shartni qanoatlantiradi. Demak, $\psi \pm \varphi$ funktsiyalarning kamida bittasi $G(\lambda)$ ning -1 xos qiymatiga mos xos funktsiyasi bo'ladi va shuning uchun umumiylikni buzmasdan $G(\lambda)$ operator $|\tilde{\psi}(\cdot)|$ juft funktsiya bo'lgan $\tilde{\psi}$ xos funktsiyaga ega deb faraz qilishimiz mumkin.

$d = 3,4$ o'lchamlarda asosiy natijani olish uchun bizga qo'shimcha $\mathbb{T}^d$ dagi silliq (davriy) funktsiyalar fazosining

$$\|f\|_\mu = \sup_{t,\ell\in\mathbb{T}^d} \Big[|f(t)| + |\ell|^{-\mu}|f(t+\ell) - f(t)|\Big].$$

normaga nisbatan yopig'i yordamida hosil qilinuvchi $\mathcal{B}(\mu)$, $0 < \mu \le 1$ Banach fazosida aniqlangan $\mathbb{T}^d$ dagi Gyoldir uzluksiz funktsiyalariga zarurat bor.([61] ga q.)

$\mathcal{B}(\mu)$ fazolar bir-biriga tabiiy

$$\mathcal{B}(\nu) \subset \mathcal{B}(\mu) \subset C(\mathbb{T}^d), \quad 0 < \mu \le \nu \le 1$$

ravishda joylashtirilganligini ta'kidlaymiz.

Agar $d = 3$ uchun $v \in \mathcal{B}(\kappa)$ va $\kappa > \frac{1}{2}$ hamda $d = 4$ uchun $\kappa > 0$ bo'lsa, Birman-Schwinger prinsipining quyidagi ko'rinishi h ning muhim spektri bo'sag'asi bo'lgan $\varepsilon(0)$ soni h operator uchun bo'sag'a xos qiymati yoki bo'sag'a sathi bo'lib xizmat qiladi.

Tasdiq 3.10. *([61] bilan taqqoslang) Faraz qilaylik $d = 3,4$, $v \in \mathcal{B}(\kappa)$ va 3.1 talab o'rinli bo'lsin.Bunda*

$$\kappa > \begin{cases} \frac{1}{2}, & \text{agar } d = 3 \\ 0, & \text{agar } d = 4. \end{cases} \tag{3.8}$$

U holda $h - \varepsilon(0)I$ operator faqat va faqat -1 soni $G(\varepsilon(0))$ operatorning xos qiymati bo'lganda va mos xos funktsiyalardan biri ψ

$$\psi(0) = 0$$

shartni qanoatlantirganda notrivial yadroga ega bo'ladi.

Xususan h operator bo'sag'a sathiga ega bo'lishi uchun -1 soni $G(\varepsilon(0))$ operatorning xos qiymati bo'lishi va unga mos xos funktsiyalardan biri ψ

$$\psi(0) \neq 0$$

shartni qanoatlantirishi zarur.

Proof. Umumiylikni buzmasdan $\varepsilon(0) = 0$ deb faraz qilamiz.
"Zarurligi." $d = 3,4$ bo'lsin. F.q. $f \in L^2(\mathbb{T}^d)$ funktsiya $h(0)$ operatornning nol xos qiymatiga mos xos funktsiyasi bo'lsin, ya'ni

$$-\varepsilon(p)f(p) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(p-q)f(q)dq, \tag{3.9}$$

tenglama deyarli barcha $p \in \mathbb{T}^d$ larda no'lmas yechimga ega.

Xuddi Lemma 3.2 ning isbotidagidek tasdiq ko'rsatadiki, f funktsiyalar sinfida unga ekvivalent shunday $\tilde{f}$ element topiladiki,

$$\psi(p) = \varepsilon(p)\tilde{f}(p) \tag{3.10}$$

tenglik bajariladi va $\psi \in \mathcal{B}(\kappa)$ Hölder uzluksiz funktsiyasi bo'ladi.

Shunday qilib, $\tilde{f}$ element koordinatalar boshidan tashqarida uzluksiz va 3.1 talabga ko'ra $\lim\inf_{p\to 0} \varepsilon(p)|p|^{-2} > 0$ bo'lganligi uchun quyidagi asimptotik yoyilma o'rinli

$$\tilde{f}(p) = \frac{\psi(0)}{\varepsilon(p)} + \mathcal{O}(|p|^{-2+\kappa}), \quad p \to 0.$$

$\tilde{f} \in L^2(\mathbb{T}^3)$ va (3.8) munosabatlardan ψ Hölder uzluksiz funktsiyasi koordinatalar boshida nolga aylanishi, ya'ni $\psi(0) = 0$ kelib chiqadi.

(3.9) va (3.10) larni taqqoslab, $d = 3, 4$ bo'lgan holda -1 soni $G(\varepsilon(0))$ operatorning $C(\mathbb{T}^d)$ dan olingan $\psi(0) = 0$ tenglikni qanoatlantiruvchi ψ xos funktsiyaga mos xos qiymati bo'ladi degan xulosaga kelamiz.

"Yetarliligi." Faraz qilaylik, $G(\varepsilon(0))$ operator $\lambda = -1$ xos qiymatga ega bo'lsin va

$$G(\varepsilon(0))\psi = -\psi \tag{3.11}$$

tenglama yechimi bo'lgan ψ xos funktsiya uchun $\psi(0) = 0$ o'rinli bo'lsin. 3.2 lemma isbotidagi mulohazalardan $\psi \in \mathcal{B}(\kappa)$ ekanligi kelib chiqadi.

$$f(p) = \frac{\psi(p)}{\varepsilon(p)}, \quad p \neq 0$$

funktsiyani kiritamiz. Ravshanki, yuqoridagi tasdiqdan quyidagi

$$f(p) = \mathcal{O}(|p|^{-2+\kappa}), \quad p \to 0$$

asimptotik yoyilma o'rinli ekanligi ko'rinadi. (3.8) ni hisobga olsak $f \in L^2(\mathbb{T}^d)$ ekanligi isbot bo'ladi va u holda (3.11) tenglikdan $h(0)$operator notrivial yadroga ega degan xulosaga kelamiz. Teorema isbot bo'ldi.

□

4. Ikki zarrachali Hamiltonian va uni bir zarrachali holga keltirish

4.1. **Koordinata tasviri.** Bo'lim davomida $d \geq 1$ deb faraz qilamiz. Mos ravishda $\varepsilon_\alpha(p), \alpha = 1, 2,$ dispersion munosabatli ikki kvant zarrachalar sistemasining erkin Hamiltoniani $\widehat{H}^0$ ($\ell^2((\mathbb{Z}^d)^2) \simeq \ell^2(\mathbb{Z}^d) \otimes \ell^2(\mathbb{Z}^d)$ Hilbert fazosidagi o'z-o'ziga qo'shma chegaralangan operator sifatida)

$$\widehat{H}^0 = \hat{h}_1^0 \otimes I + I \otimes \hat{h}_2^0, \tag{4.1}$$

kabi kiritiladi, bunda

$$\hat{h}_\alpha^0 = \varepsilon_\alpha(-\mathrm{i}\nabla), \quad \alpha = 1, 2,$$

va $I-$ $\ell^2(\mathbb{Z}^d)$ dagi birlik operator.

O'zaro ta'sir potensiali (koordinata tasvirida) $\widehat{V}$ haqiqiy qiymatli bo'lgan ikki zarrachali sistema to'la Hamiltoniani $\widehat{H}$ $\ell^2((\mathbb{Z}^d)^2)$ Hilbert fazosida o'z-o'ziga qo'shma chegaralangan operator bo'lib, quyidagi ko'rinishga ega:

$$\widehat{H} = \widehat{H}^0 + \widehat{V},$$

bunda

$$(\widehat{V}\hat{\psi})(x_1, x_2) = \hat{v}(x_1 - x_2)\hat{\psi}(x_1, x_2), \quad \hat{\psi} \in \ell^2((\mathbb{Z}^d)^2),$$

$v(p)$ uzluksiz funktsiyaning $\{\hat{v}(s)\}_{s\in\mathbb{Z}^d}$ Fourier koeffitsientlari 3.1 talabni qanoatlantiradi.

4.2. **Impuls tasvirga otish.** Impuls tasvirga

$$\mathcal{F}_2 : L^2((\mathbb{T}^d)^2) \longrightarrow \ell^2((\mathbb{Z}^d)^2),$$

standard Fourier almashtirishi orqali o'tiladi, bunda $(\mathbb{T}^d)^m$ orqali uch o'lchamli $\mathbb{T}^d = (-\pi, \pi]^d$ kubning m-darajali Dekart ko'paytmasi belgilangan:

$$(\mathbb{T}^d)^m = \underbrace{\mathbb{T}^d \times \mathbb{T}^d \times \cdots \times \mathbb{T}^d}_{m \text{ marta}}, \quad m \in \mathbb{N}. \tag{4.2}$$

Ikki zarrachali H Hamiltonianning impuls ko'rinishi

$$H = H^0 + V$$

kabi bo'ladi, bunda

$$(H^0 f)(k_1,k_2) = (\varepsilon_1(k_1)+\varepsilon_2(k_2))f(k_1,k_2), \quad f \in L^2((\mathbb{T}^d)^2),$$

va V xususiy integrallash operatori

$$(Vf)(k_1,k_2) = (2\pi)^{-\frac{d}{2}} \int\limits_{(\mathbb{T}^d)^2} v(k_1-k_1')\delta(k_1+k_2-k_1'-k_2')f(k_1',k_2')dk_1'dk_2',$$

$$f \in L^2((\mathbb{T}^d)^2)$$

kabidir.

Bunda yadro funktsiyasi

$$v(p) = (2\pi)^{-d/2} \sum_{s\in\mathbb{Z}^d} \hat{v}(s)\, e^{\mathrm{i}(p,s)}, \quad p \in \mathbb{T}^d,$$

Fourier qatori orqali beriladi va $\delta(p)$ orqali Dirakning delta-funktsiyasi belgilanadi.

4.3. **To'g'ri integralga yoyish. Kvaziimpuls.** $\hat{U}_s^2$, $s \in \mathbb{Z}^d$, orqali Abel guruhi $\mathbb{Z}^d$ -ning Hilbert fazosi $\ell^2((\mathbb{Z}^d)^2)$ da siljitish (unitar) operatorlari orqali tasvirini belgilaymiz:

$$(\hat{U}_s^2\hat{\psi})(n_1,n_2) = \hat{\psi}(n_1+s, n_2+s), \quad \hat{\psi} \in \ell^2((\mathbb{Z}^d)^2), \quad n_1,n_2,s \in \mathbb{Z}^d.$$

$\mathcal{F}_2$ Fourier almashtirishi

$$\hat{U}_{s+t}^2 = \hat{U}_s^2\hat{U}_t^2, \quad s,t \in \mathbb{Z}^d,$$

$\mathbb{Z}^d$ abel guruhining Hilbert fazosi $\ell^2((\mathbb{Z}^d)^2)$ dagi tasvirini ushbu $U_s^2 = \mathcal{F}_2^{-1}\hat{U}_s^2\mathcal{F}_2$, $s \in \mathbb{Z}^d$ unitar (ko'paytirish) operatorlari orqali

$$(U_s^2 f)(k_1,k_2) = \exp\big(-\mathrm{i}(s,k_1+k_2)\big)f(k_1,k_2), \quad k_1,k_2 \in \mathbb{T}^d, \quad f \in L^2((\mathbb{T}^d)^2) \tag{4.3}$$

tasvirga keltiradi.

Berilgan $k \in \mathbb{T}^d$ uchun, $\mathbb{F}_k$ ni quyidagicha aniqlaymiz:

$$\mathbb{F}_k = \{(k_1, k-k_1) \in (\mathbb{T}^d)^2 : k_1 \in \mathbb{T}^d,\ k-k_1 \in \mathbb{T}^d\}.$$

$$\pi : (\mathbb{T}^d)^2 \to \mathbb{T}^d, \quad \pi((k_1,k_2)) = k_1,$$

akslantirishni kiritamiz, π_k, $k \in \mathbb{T}^d$ orqali esa π ning $\mathbb{F}_k \subset (\mathbb{T}^d)^2$ dagi qismini belgilaymiz, ya'ni

$$\pi_k = \pi|_{\mathbb{F}_k}. \tag{4.4}$$

$\mathbb{F}_k$, $k \in \mathbb{T}^d$ ko'pxillik $\mathbb{T}^d$ ga gomeomorf ekanligini eslatib o'tamiz.

Quyidagi lemma o'rinli.

Lemma 4.1. *Teskarisi*

$$(\pi_k)^{-1}(q) = (q, k-q)$$

ko'rinishida aniqlangan $\mathbb{F}_k \subset (\mathbb{T}^d)^2$ *ni* $\mathbb{T}^d$ *ning ustiga akslantiruvchi* π_k, $k \in \mathbb{T}^d$ *akslantirish biyektivdir.*

$L^2((\mathbb{T}^d)^2)$ Hilbert fazosini

$$L^2((\mathbb{T}^d)^2) = \int_{k\in\mathbb{T}^d} \oplus L^2(\mathbb{F}_k)dk$$

to'g'ri integralga yoyib U_s^2, $s \in \mathbb{Z}^d$ unitar yoyilmaga mos

$$U_s^2 = \int_{k\in\mathbb{T}^d} \oplus U_s(k)dk,$$

to'g'ri integral yoyilmani hosil qilamiz, bunda

$$U_s(k) = e^{-\mathrm{i}(s,k)} I_{L^2(\mathbb{F}_k)}$$

va $I_{L^2(\mathbb{F}_k)}$- $L^2(\mathbb{F}_k)$ Hilbert fazosidagi birlik operator ..

Ravshanki, $\widehat{H}$ (koordinata tasvirida) Hamiltonian $\hat{U}_s^2$, $s \in \mathbb{Z}^d$ guruh bilan o'rin almashinuvchi, ya'ni

$$\hat{U}_s^2\widehat{H} = \widehat{H}\hat{U}_s^2, \quad s \in \mathbb{Z}^d.$$

Xuddi shunday H (impuls tasvirda) Hamiltonian ham (10.3) formula bilan aniqlangan U_s^2, $s \in \mathbb{Z}^d$ guruh bilan o'rin almashinuvchidir. Demak, H operatorni

$$L^2((\mathbb{T}^d)^2) = \int_{k\in\mathbb{T}^d} \oplus L^2(\mathbb{F}_k)dk.$$

yoyilmaga mos

$$H = \int_{k\in\mathbb{T}^d} \oplus \tilde{h}(k)dk \tag{4.5}$$

to'g'ri integralga yoyish mumkin.

Bunda k, $k \in \mathbb{T}^d$ parametrga *ikki zarrachali kvaziimpuls* va unga mos $\tilde{h}(k),\ k \in \mathbb{T}^d$ operatorlarga *qobiq operatorlar* deyiladi.

4.4. **Ikki zarrachali dispersion munosabatlar.** (4.5) yoyilmadagi $\tilde{h}(k),\ k \in \mathbb{T}^d$ operatorlar

$$h(k) = h^0(k) + v$$

ko'rinishdagi $h(k)$, $k \in \mathbb{T}^d$ operatorlarga unitar ekvivalent, bunda

$$(h^0(k)f)(p) = \mathcal{E}_k(p)f(p),$$

$$(vf)(p) = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{T}^d} v(p-q)f(q)dq, \quad f \in L^2(\mathbb{T}^d)$$

va

$$\mathcal{E}_k(p) = \varepsilon_1(p) + \varepsilon_2(k-p), \qquad p \in \mathbb{T}^d,$$

ikki zarrachali dispersion munosabatlar k, $k \in \mathbb{T}^d$ kvaziimpulsga parametrik bog'liq.

Ekvivalentlik $u_k : L^2(\mathbb{F}_k) \to L^2(\mathbb{T}^d),\ k \in \mathbb{T}^d$,

$$u_k g = g \circ (\pi_k)^{-1},$$

unitar operator orqali quriladi, bunda π_k (10.5) formula bilan aniqlangan.

5. $h(k)$ QOBIQ OPERATORLARNING SPEKTRAL XOSSALARI

Oldingi bo'limda o'rganganimizdek, ikki zarrachali H Hamiltonian

$$H \simeq \int\limits_{k\in\mathbb{T}^d} \oplus h(k)dk$$

to'g'ri integralga yoyiladi.

Bu

$$h(k) = h^0(k) + v$$

qobiq operatorlarni

$$\mathcal{E}_k(p) = \varepsilon_1(p) + \varepsilon_2(k-p), \qquad p \in \mathbb{T}^d, \tag{5.1}$$

ikki zarrachali dispersion munosabatli bir zarrachali Hamiltonian sifatida qarash mumkin. Bunda $\varepsilon_\alpha(p)$ $\alpha = 1,2$ funktsiyalar zarrachalarning dispersion munosabatlarni ifodalaydi.

3.1 talabga ko'ra, $h^0(k)$, $k \in \mathbb{T}^d$ operatorning v qo'zg'alishi Hilbert-Schmidt operatoridir va shuning uchun Weyl teoremasiga ko'ra $h(k)$ operatorning muhim spektri haqiqiy o'qning quyidagi kesmasidan iborat:

$$\sigma_{\text{ess}}(h(k)) = [\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)],$$

bunda

$$\mathcal{E}_{\min}(k) = \min_{q\in\mathbb{T}^d} \mathcal{E}_k(q), \quad \mathcal{E}_{\max}(k) = \max_{q\in\mathbb{T}^d} \mathcal{E}_k(q).$$

Agar bir zarrachali qism dispersion munosabatlar shartli manfiy aniqlangan bo'lsa, u holda k kvaziimpulsning nol qiymatiga mos ikki zarrachali $\mathcal{E}_0(p)$ dispersion munosabat ham shartli manfiy aniqlangan bo'ladi. Demak bu shartlarga ko'ra $h(0)$ Hamiltonian koordinata ko'rinishida $e^{-th(0)}$, $t > 0$ yarim guruhga nisbatan musbatlikni saqlaydi.

(Umuman olganda qobiq Hamiltonian $h(k)$,$k \neq 0$ lar uchun bu tasdiq o'rinli emas, ya'ni $\mathcal{E}_k(p)$ funktsiya juft bo'lmasligi va shuning uchun shartli manfiy aniqlanmagan bo'lishi mumkin).

Ikki zarrachali dispersion munosabat shartli manfiy aniqlanmasada, kvaziimpulsning notrivial qiymatlari uchun ular yana ba'zi muhim kerakli tengsizliklar, ya'ni bir zarrachali dispersion munosabat uchun ifodalangan 2.5 Lemmaning o'xshashi bo'lgan quyida keltirilgan 5.2 Lemmani qanoatlantiradi.

Talab 5.1. *Faraz qilaylik $\varepsilon_1(p)$ va $\varepsilon_2(p)$ larning har ikkalasi uchun ham 3.1 talab o'rinli va bu $\varepsilon_\alpha(p)$, $\alpha = 1,2$ bir zarrachali dispersion munosabatlar shartli manfiy aniqlangan bo'lsin.*

Agar $\varepsilon_1(p)$ va $\varepsilon_2(p)$ lar 3.1 talabni qanoatlantirsa, $\mathcal{E}_0(p)$ ikki zarrachali dispersion munosabat ham bu talabni qanoatlantiradi.

Lemma 5.2. *Faraz qilaylik 5.1 talab o'rinli bo'lsin. U holda $k \neq q$ yoki $q \neq 0$ bo'lgan ixtiyoriy (mahkamlangan) $k, q \in \mathbb{T}^d$ lar va deyarli barcha $p \in \mathbb{T}^d$ lar uchun*

$$\mathcal{E}_0(p) - \mathcal{E}_0(0) + \mathcal{E}_k(q) - \frac{\mathcal{E}_k(p+q) + \mathcal{E}_k(q-p)}{2} > 0,$$

o'rinli bo'ladi.

Xususan, agar $k \neq 0$ va biror $p(k)$ nuqtada $\mathcal{E}_k(\cdot)$ funktsiya o'zining minimum qiymatiga erishsa, ya'ni

$$\mathcal{E}_{\min}(k) = \mathcal{E}_k(p(k)),$$

u holda deyarli barcha $p \in \mathbb{T}^d$ lar uchun quyidagi

$$\mathcal{E}_0(p) - \mathcal{E}_{\min}(0) + \mathcal{E}_{\min}(k) - \frac{\mathcal{E}_k(p+p(k)) + \mathcal{E}_k(p(k)-p)}{2} > 0,$$

tengsizlik o'rinli bo'ladi.

Proof. $|q|^2 + |k-q|^2 \neq 0$ bo'lganligi uchun bu tasdiq 2.5 Lemmaning bevosita natijasi bo'ladi va ikki zarrachali dispersion munosabatning (5.1) aniqlanishidan deyarli barcha $p \in \mathbb{T}^d$ lar uchun

$$\begin{aligned}
&\mathcal{E}_0(p) + \mathcal{E}_k(q) - \frac{\mathcal{E}_k(p+q) + \mathcal{E}_k(q-p)}{2} - \mathcal{E}_0(0) \\
&= \varepsilon_1(p) + \varepsilon_1(q) - \frac{\varepsilon_1(p+q) + \varepsilon_1(q-p)}{2} - \varepsilon_1(0) \\
&+ \varepsilon_2(p) + \varepsilon_2(k-q) - \frac{\varepsilon_2(k-q-p) + \varepsilon_2(k-q+p)}{2} - \varepsilon_2(0) > 0,
\end{aligned}$$

tengsizlik o'rinli bo'ladi. □

Bundan keyin quyida $d \geq 3$ deb faraz qilamiz.

Bizning birinchi asosiy natijamiz 5.1 talab bjarilganda $h(k)$ qobiq operatorning diskret spektri kvaziimpuls o'zgarganda bo'sag'a orqali yutulishi(yoqolishi) mumkin emasligini korsatadi.

Teorema 5.3. *Faraz qilaylik 5.1 talab bajarilsin va $\varepsilon_j(p)$, $j = 1, 2$ dispersion munosabatlar ikki marta differensiallanuvchi funktsiyalar bo'lsin.*

$m(k)$, $k \in \mathbb{T}^d$ orqali $h(k)$ operatori spektrinig quyi chegarasini belgilaymiz:

$$m(k) = \inf \operatorname{spec}(h(k)), \quad k \in \mathbb{T}^d.$$

$h(0)$ operator spektrining quyi chegarasi $m(0) = \mathcal{E}_{\min}(0)$ xos qiymat bo'lsin. U holda

$$0 \leq \mathcal{E}_{\min}(0) - m(0) < \mathcal{E}_{\min}(k) - m(k), \quad k \in \mathbb{T}^d,\ k \neq 0, \quad d \geq 3 \tag{5.2}$$

o'rinlidir.

Proof. $0 \neq f \in \operatorname{Ker}(h(0) - m(0)I)$ bo'lsin. Bundan deyarli barcha $p \in \mathbb{T}^d$ lar uchun

$$\mathcal{E}_0(p) f(p) + (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(p-q) f(q) dq = m(0) f(p)$$

tenglikka ega bo'lamiz.

Qo'yilgan shartlarga ko'ra bir zarrachali dispersion munosabatlar shartli manfiy aniqlangan funktsiyalar bo'ladi.

U holda ikki zarrachali dispersion munosabatning aniqlanishidan, k kvaziimpulsning nol qiymatiga mos $\mathcal{E}_0(p)$ funktsiya ham shartli manfiy aniqlangan. Xususan, $\mathcal{E}_0(p)$ juft funktsiya va shuning uchun 3.9 Eslatmaga ko'ra, umumiylikni buzmasdan $|f(\cdot)|$ funktsiyani juft deb faraz qilishimiz mumkin.

$k \in \mathbb{T}^d$ uchun $L^2(\mathbb{T}^d)$ da quyidagi (asosiy)

$$f_k(p) = f(p - p(k))$$

funktsiyani kiritamiz, bunda $p(k)$ orqali $\mathcal{E}_k(p)$ funktsiyaning minimum nuqtasi belgilangan, ya'ni $\mathcal{E}_k(p(k)) = \mathcal{E}_{\min}(k)$ ($\mathcal{E}_k(p)$ o'zining minimum qiymatiga har biri ixtiyoriy tanlangan $p(k)$ ning bir nechta qiymatlarida erishishi mumkin).

(5.2) ni isbotlash uchun

$$\Gamma(k) = ([h(k) - (\mathcal{E}_{\min}(k) - \mathcal{E}_{\min}(0) + m(0))] f_k, f_k) < 0 \quad k \neq 0 \tag{5.3}$$

tengsizlikni ko'rsatishimiz kerak.

Ma'lumki

$$
\begin{aligned}
(5.4) \qquad & ([h(k) - (\mathcal{E}_{\min}(k) - \mathcal{E}_{\min}(0) + m(0))]f_k)(p) \\
&= [\mathcal{E}_k(p) - (\mathcal{E}_{\min}(k) - \mathcal{E}_{\min}(0) + m(0))]f(p - p(k)) \\
&+ (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(p-q) f(q - p(k)) dq \\
&= [\mathcal{E}_k(p) - (\mathcal{E}_{\min}(k) - \mathcal{E}_{\min}(0) + m(0))]f(p - p(k)) \\
&+ (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(p - p(k) - q) f(q) dq \\
&= [\mathcal{E}_k(p) - \mathcal{E}_{\min}(k) - \mathcal{E}_0(p - p(k)) + \mathcal{E}_{\min}(0)]f(p - p(k)).
\end{aligned}
$$

(5.4) dan foydalansak,

$$
\Gamma(k) = -\int\limits_{\mathbb{T}^d} \Big(\mathcal{E}_0(p - p(k)) - \mathcal{E}_{\min}(0) - \mathcal{E}_k(p) + \mathcal{E}_{\min}(k)\Big)|f(p - p(k))|^2 dp, \tag{5.5}
$$

$$
k \in \mathbb{T}^d
$$

yoyilma hosil bo'ladi.

(7.8) da $p \to -p + 2p(k)$ o'zgaruvchini almashtirib, hamda $\mathcal{E}_0(p)$ va $|f(p)|$ funktsiyalarning juftligidan foydalanib,

$$
\Gamma(k) = -\int\limits_{\mathbb{T}^d} \big(\mathcal{E}_0(p - p(k)) - \mathcal{E}_{\min}(0) - \mathcal{E}_k(-p + 2p(k)) + \mathcal{E}_{\min}(k)\big)|f(p - p(k))|^2 dp \tag{5.6}
$$

tenglikka ega bo'lamiz.

(7.8) hamda (5.6) da yana bir marta $p \to p - p(k)$ o'zgaruvchini almashtirib va natijalarni qo'shib

$$
\Gamma(k) = -\int\limits_{\mathbb{T}^d} \mathcal{F}(k, p)|f(p)|^2 dp
$$

ga kelamiz, bunda

$$
\mathcal{F}(k, p) = \mathcal{E}_0(p) - \mathcal{E}_{\min}(0) + \mathcal{E}_{\min}(k) - \frac{\mathcal{E}_k(p + p(k)) + \mathcal{E}_k(p(k) - p)}{2}.
$$

5.2 lemmaga ko'ra, ixtiyoriy (mahkamlangan) $k \neq 0$ uchun, deyarli barcha $p \in \mathbb{T}^d$ larda $\mathcal{F}(k, p) > 0$ ekanligi kelib chiqadi. Bu esa yuqoridagi (5.3) tengsizlikni isbotlaydi va natijada tasdiq ham isbotlanadi.

□

Bizning ikkinchi asosiy natijamiz $k \neq 0$ da $h(k)$ Hamiltonianlar oilasining diskret spektri mavjudligi uchun yetarlilik shartini aniqlashdir.

Teorema 5.4. *5.1 talab o'rinli va $\varepsilon_j(p)$, $j = 1, 2$ dispersion munosabatlar ikki marta differensiallanuvchi funktsiyalar bo'lsin. Faraz qilaylik, $h(0)$ operator yoki bo'sag'a xos qiymatga yoki virtual sathga ega bo'lsin.*

U holda barcha $k \in \mathbb{T}^d \setminus \{0\}$ larda $h(k)$ Hamiltonianning muhim spektri tubi $\mathcal{E}_{\min}(k)$ dan quyida joylashgan diskret spektri to'plami bo'sh emas.

Proof. $h(0)$ bo'sag'adan pastda yoki $m(0)$ muhim spektr tubidan quyida diskret spektrga ega bo'lgan holda

$$
m(0) = \mathcal{E}_{\min}(0) = \mathcal{E}_0(0) \tag{5.7}
$$

(bo'sag'a) xos qiymat bo'lishi Theorem 5.3 da (allaqachon) isbotlangan.

Faraz qilaylik, $d = 3$ yoki $d = 4$ va $h(0)$ uning muhim spektri tubida virtual sathga ega bo'lsin. Demak,

$$
G(\mathcal{E}_0(0))\psi = -\psi, \quad \psi \in C(\mathbb{T}^d),
$$

tenglama $\psi \in C(\mathbb{T}^d)$ nolmas yechimga ega. 5.3 teoremada da isbotlanganidek, umumiylikni buzmasdan $|\psi(p)|$ funktsiya juft deb faraz qilish mumkin. Xususan, deyarli barcha $p \in \mathbb{T}^d$ larda

$$
\mathcal{E}_0(p) f(p) + (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(p - q) f(q) dq = m(0) f(p), \tag{5.8}
$$

larda tenglama $L^1(\mathbb{T}^d)$ da shunday

$$
f(p) = \frac{\psi(p)}{\mathcal{E}_0(p) - \mathcal{E}_0(0)}
$$

yechimga egaki,(3.7 bilan solishtiring) $|f(\cdot)|$ funktsiya juft bo'ladi.

Berilgan $k \in \mathbb{T}^d$ uchun $L^2(\mathbb{T}^d)$ fazoda $\{f_{n,k}\}_{n=1}^{\infty}$ funktsiyalar ketma-ketligi

$$f_{n,k}(p) = \frac{\psi(p - p(k))}{\mathcal{E}_0(p - p(k)) - \mathcal{E}_0(0) + \frac{1}{n}}$$

ni qaraymiz.

Lebegning yaqinlashish teoremasiga asosan $f_{n,k}$ ketma-ketlik $n \to \infty$ da $L^1(\mathbb{T}^d)$ fazoning f_k

$$f_k(p) = f(p - p(k)), \quad p \in \mathbb{T}^d,$$

elementiga yaqinlashadi, bunda $f(\cdot)$ majorant integrallanuvchi.

5.1 talabdan $[h(k) - \mathcal{E}_{\min}(k)]f_{k,n}$ funktsiyalar ketma-ketligi $L^\infty(\mathbb{T}^d)$ dagi norma bo'yicha chegaralanganligi kelib chiqadi. Agar (5.7), (5.8) va

$$\begin{aligned}&([h(k) - \mathcal{E}_{\min}(k)]f_{k,n})(p)\\ &= [\mathcal{E}_k(p) - \mathcal{E}_{\min}(k)]f_{k,n}(p) + (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(p-q) f_{k,n}(q)dq\end{aligned}$$

tenglikdan foydalansak

$$\begin{aligned}&[\mathcal{E}_k(p) - \mathcal{E}_{\min}(k)]f_k(p) + (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(p-q) f_k(q)dq\\ &= [\mathcal{E}_k(p) - \mathcal{E}_{\min}(k) - \mathcal{E}_0(p - p(k)) + \mathcal{E}_0(0)]f(p - p(k))\end{aligned}$$

ga kelamiz.

Xususan,

$$\Gamma(k) = \lim_{n\to\infty}([h(k) - \mathcal{E}_{\min}(k)]f_{n,k}, f_{n,k}), \quad k \in \mathbb{T}^d,$$

limit mavjud va chekli, bundan tashqari

$$\Gamma(k) = -\int_{\mathbb{T}^d} \frac{\mathcal{E}_0(p - p(k)) - \mathcal{E}_0(0) - \mathcal{E}_k(p) + \mathcal{E}_{\min}(k)}{(\mathcal{E}_0(p - p(k)) - \mathcal{E}_0(0))^2} |\psi(p - p(k))|^2 dp, \quad k \in \mathbb{T}^d.$$

Yuqrida keltirilgan 5.3 Teoremaning isbotidagi kabi mulohazalar bilan

$$\Gamma(k) < 0, \quad k \neq 0. \tag{5.9}$$

tengsizlik bajarilishini tekshirish mumkin.

(5.9) tengsizlikka ko'ra shunday $n_0 \in \mathbb{N}$ mavjudki,

$$([h(k) - \mathcal{E}_{\min}(k)]f_{n_0,k}, f_{n_0,k}) < 0, \quad k \neq 0$$

bajariladi.

Bu esa $k \neq 0$ uchun $h(k)$ ning muhim spektri tubidan pastda discret spektrining mavjudligini isbotlaydi.

□

Eslatma 5.5. *$d = 3$ bo'lsin. $h(k)$ Hamiltonian muhim spektrining qalinligi $w(k)$*

$$w(k) = \mathcal{E}_{\max}(k) - \mathcal{E}_{\min}(k)$$

$k \in \mathbb{T}^3$ kvaziimpulsning ba'zi qiymatlarida yo'qolishi mumkin. Demak, $h(k)$ qobiq Hamiltonian hattoki, agar $h(0)$ ning diskret spektri bo'sh to'plam bo'lsa ham k kvaziimpulsning ba'zi qiymatlarida cheksiz ko'p diskret spektrga ega bo'lishi mumkin. Masalan $\mathbb{Z}^3$ panjarada

$$\varepsilon_1(p) = \varepsilon_2(p) = \sum_{i=1}^{3}(1 - \cos p_i), \quad p \in \mathbb{T}^3 \tag{5.10}$$

bir zarrachali dispersion munosabatli ikki zarrachani qaraymiz.

U holda agar $k_0 = (\pi, \pi, \pi) \in \mathbb{T}^3$ bo'lsa, barcha $p = (p_1, p_2, p_3) \in \mathbb{T}^3$ lar uchun o'rinli bo'lgan

$$\mathcal{E}_{k_0}(p) = \varepsilon_1(p) + \varepsilon_2(k_0 - p) = 6$$

ikki zarrachali dispersion munosabatning "kuchli aynishi"ni ko'ramiz.

Bu $h(k_0)$ ning muhim spektri bitta nuqtadan iborat to'plam, xususan $\mathrm{spec}_{ess}(h(k_0)) = \{6\}$ ekanligini anglatadi.

Natijada $\hat{h}(k_0) = \mathcal{E}_{k_0}(-\mathrm{i}\nabla) + \hat{v}$ qobiq Hamiltoniani o'zining muhim spektri tubidan pastda

(i) $\hat{v} = \{\hat{v}(s)\}_{s\in\mathbb{Z}^3} \subset \ell^1(\mathbb{Z}^3)$,
(ii) $\#\{s \in \mathbb{Z}^3 \,|\, \hat{v}(s) < 0\} = \infty$,

shartlarni qanoatlantiruvchi cheksiz ko'p diskret spektrga ega.

Agar qo'shimcha ravishda

(iii) *(uzluksiz)* $v(p) = (2\pi)^{-\frac{3}{2}} \sum_{s\in\mathbb{Z}^3} \hat{v}(s) e^{\mathrm{i}(p,s)}$

funktsiyaning $C(\mathbb{T}^3)$ *dagi normasi yetarlicha kichikligi talab qilinsa, u holda kvaziimpulsning nol qiymatiga mos* $\hat{h}(0) = \mathcal{E}_0(-\mathrm{i}\nabla) + \hat{v}$ *Hamiltonianning diskret spektri bo'sh to'plam bo'ladi (ya'ni yadro funktsiyasi* (3.2) *orqali berilgan* $C(\mathbb{T}^3)$ *dagi* $G(\lambda)$ *Birman-Schwinger integral operatori*

$$G(p,q;\lambda) = (2\pi)^{-\frac{3}{2}} v(p-q)(\mathcal{E}_0(q)-\lambda)^{-1}, \quad p,q\in\mathbb{T}^3,$$
$$\lambda \notin [\mathcal{E}_{\min}(0), \mathcal{E}_{\max}(0)],$$

$\|v\|_{C(\mathbb{T}^3)}$ *yetarlicha kichik bo'lganda torayish siqish operatori bo'ladi)*

Yana shuni eslatish muhimki, kvaziimpulsning ba'zi $k \neq 0$ *qiymatlarida ikki zarrachali* $\mathcal{E}_k(\cdot)$ *dispersion munosabatning (qisman aynishi)* $h(k)$ *Hamiltonianning* $[\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)]$ *kesma (oraliq)dan tashqarida cheksiz ko'p diskret spektirni paydo qilishi mumkin.*

Eslatma 5.6. *Muhim spektr tepasidan yuqorida qobiq Hamiltonianlarning diskret spektrini o'rganish ko'rsatadiki yuqorida tasvirlangan yo'naltiruvchi chiziqlar, bir zarrachali* $\varepsilon_\alpha(p)$, $\alpha = 1,2$ *dispersion munosabatlar shartli manfiy aniqlangan funktsiyalar bo'lishiga imkon yaratadi*

6. Bo'sag'a xos qiymatining va virtual sathning mavjudligi

Bu ilovaning asl maqsadi qurilgan misol orqali IV holning bo'sh emasligini ko'rsatishdir.

Misol 6.1. $h_{\lambda,\mu},\ \lambda,\mu\in\mathbb{R}$,

$$\hat{h}_{\lambda,\mu} = -\Delta + \hat{v}_{\lambda,\mu},$$

ko'rinishidagi diskret Schrödinger operatori bo'lsin, bunda Δ *2.4 misoldagi diskret Laplas operatori va*

$$\hat{v}_{\lambda,\mu}(s) = \begin{cases} \mu, & s=0 \\ \frac{\lambda}{2}, & |s|=1 \\ 0, & \textit{qolgan hollarda}. \end{cases}$$

Bu holda o'zaro ta'sirning Fourier almashtirishi

$$v(p) = \frac{1}{(2\pi)^{\frac{3}{2}}}\left(\mu + \lambda\sum_{i=1}^{3}\cos p_i\right)$$

to'liq hisoblanadi va natijada Birman-Schwinger yadrosi uchun

$$G(p,q;0) = \frac{1}{(2\pi)^3}\frac{\mu + \lambda\sum_{i=1}^3 \cos(p_i-q_i)}{\varepsilon(q)}, \quad p,q\in\mathbb{T}^3 \tag{6.1}$$

yoyilma kelib chiqadi, bunda $\varepsilon(q)$ (2.4) formula bilan aniqlangan va $\varepsilon(0)=0$.

Quyidagicha

$$\begin{aligned} a &= \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\frac{dq}{\varepsilon(q)}, & c &= \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\frac{\cos q_i dq}{\varepsilon(q)}, \quad i=1,2,3,\\ s &= \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\frac{\sin^2 q_i dq}{\varepsilon(q)}, & b &= \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\frac{\cos^2 q_i dq}{\varepsilon(q)}, \quad i=1,2,3,\\ d &= \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\frac{\cos q_i \cos q_j dq}{\varepsilon(q)}, & & \qquad i,j=1,2,3,\ i\neq j. \end{aligned}$$

belgilashlar kiritamiz.

Eslatib o'tamizki, $\varepsilon(q) = \varepsilon(q_1,q_2,q_3)$ funktsiya o'zining q_1, q_2 va q_3 argumentlarining o'rin almashtirishlariga nisbatan invariant bo'lgani uchun yuqoridagi c,s,b,d integrallar alohida i,j indekslarga bog'liq emas. Sodda hisoblashlar quyidagi

$$a - c = \frac{1}{6}, \tag{6.2}$$

$$b + 2d = 3c, \tag{6.3}$$

$$a = b + s \quad \text{va} \quad s = \frac{1}{6} - \frac{2}{3}(b-d) \tag{6.4}$$

munosabatlar o'nli ekanligini ko'rsatadi.

Lemma 6.2. $a > \frac{11}{51}$. *Xususan,* $c > 0$.

Proof. Isbotni

$$\int_{-\pi}^{\pi}\frac{dq}{A-\cos q}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dq}{A-\cos q}+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dq}{A+\cos q}+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dq}{A-\sin q}+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dq}{A+\sin q},\quad |A|>1, \tag{6.5}$$

yoyilma bilan boshlaymiz, bundan esa

$$a=\frac{1}{(2\pi)^3}\int_{\mathbb{T}^3}\frac{dq}{\varepsilon(q)}=\frac{1}{2(2\pi)^3}\int_{\mathbb{T}^3}\frac{dq}{3-\cos q_1-\cos q_2-\cos q_3}=\frac{1}{2(2\pi)^3}\int_{\mathbb{T}^2}dq_1dq_2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}dq_3 f(q_1,q_2,q_3), \tag{6.6}$$

ekanligi kelib chiqadi, bunda

$$f(q_1,q_2,q_3)=\frac{1}{3-\cos q_1-\cos q_2-\cos q_3}+\frac{1}{3-\cos q_1-\cos q_2+\cos q_3}+\frac{1}{3-\cos q_1-\cos q_2-\sin q_3}+\frac{1}{3-\cos q_1-\cos q_2+\sin q_3}$$

f funktsiya $(-\pi,\pi]^3\setminus\{0\}$ da aniqlanganligini ta'kidlab o'tamiz.

Mahkamlangan $q_1,\ q_2$ lar uchun $f(q_1,q_2,q_3)$ funktsiya $q_3,\ q_3\in[-\frac{\pi}{4},\frac{\pi}{4}]$ argumenti bo'yicha $[-\frac{\pi}{4},\frac{\pi}{4}]$ kesmaning chetki nuqtalarida (oxirlarida (uchlarida)) minimum qiymatga erishishini osongina tekshirish mumkin va shuning uchun

$$f(q_1,q_2,q_3)>\frac{2}{3-\cos q_1-\cos q_2-\frac{\sqrt{2}}{2}}+\frac{2}{3-\cos q_1-\cos q_2+\frac{\sqrt{2}}{2}},\quad q_3\in(-\frac{\pi}{4},\frac{\pi}{4}). \tag{6.7}$$

(6.6) va (6.7) larni umumlashtirsak:

$$a>\frac{1}{(2\pi)^3}\frac{\pi}{2}\int_{\mathbb{T}^2}dq_1dq_2\left(\frac{1}{3-\frac{\sqrt{2}}{2}-\cos q_1-\cos q_2}+\frac{1}{3+\frac{\sqrt{2}}{2}-\cos q_1-\cos q_2}\right).$$

tengsizlikka kelamiz.

(6.5) dagi usulni yana ikki marta qo'llab, (birinchi marta q_2 o'zgaruvchidan qutulish uchun va keyin q_1 ni yo'qotish uchun) isbotni yakunlovchi

$$a>\frac{1}{(2\pi)^3}\left(\frac{\pi}{2}\right)^2\int_{\mathbb{T}}dq_1\left(\frac{2}{3-2\frac{\sqrt{2}}{2}-\cos q_1}+\frac{4}{3-\cos q_1}+\frac{2}{3+2\frac{\sqrt{2}}{2}-\cos q_1}\right)$$
$$>\frac{1}{(2\pi)^3}\left(\frac{\pi}{2}\right)^3\left[\frac{4}{3-3\frac{\sqrt{2}}{2}}+\frac{12}{3-\frac{\sqrt{2}}{2}}+\frac{12}{3+\frac{\sqrt{2}}{2}}+\frac{4}{3+3\frac{\sqrt{2}}{2}}\right]=\frac{11}{51},$$

baholashga kelamiz. □

Natija 6.3.

$$\Lambda=\left\{\frac{1}{s},\frac{1}{b-d}\right\}\setminus\left\{\frac{2a}{c}\right\}$$

to'plam bo'sh emas.

Proof. $\Lambda=\emptyset$ tenglik bajarilsin deb faraz qilamiz, ya'ni

$$\frac{1}{s}=\frac{1}{b-d}=\frac{2a}{c}. \tag{6.8}$$

(6.3), (6.4) va (6.8) larni bir vaqtda yechish Lemma 6.2 ga zid bo'lgan

$$a=\frac{5}{24}<\frac{11}{51}$$

tengsizlikka olib keladi. □

Teorema 6.4. *F.q.* $-\lambda \in \Lambda$ *va*

$$\mu = -\frac{1+3\lambda c}{a+\frac{\lambda c}{2}},$$

bo'lsin.

U holda $h_{\lambda,\mu}$ *Hamiltonian virtual sathga hamda bo'sag'a xos qiymatiga ega bo'ladi.*

Proof. 3.10 xossaga ko'ra $C(\mathbb{T}^3)$ Banach fazosida (6.1) orqali berilgan $G(0)$ integral operator -1 xos qiymatga mos ikkita ψ va φ xos funktsiyalariga ega ekanligini ko'rsatishimiz zarur, ya'ni

$$G(0)\psi = -\psi \quad \text{va} \quad G(0)\varphi = -\varphi$$

bunda

$$\psi(0) \neq 0 \quad \text{va} \quad \varphi(0) = 0.$$

Barcha toq (juft) funktsiyalar fazosi $C_o(\mathbb{T}^3)$ (mos holda $C_e(\mathbb{T}^3)$) $G(0)$ integral operatorga nisbatan invariant qism fazo bo'ladi. G ning $C_o(\mathbb{T}^3)$ (mos holda $C_e(\mathbb{T}^3)$) fazodagi qismi G_o (mos holda G_e)

$$G_o(p,q) = \frac{\lambda}{(2\pi)^3}\sum_{i=1}^{3}\frac{\sin p_i \sin q_i}{\varepsilon(q)},$$

$$G_e(p,q) = \frac{1}{(2\pi)^3}\frac{\mu + \lambda\sum_{i=1}^{3}\cos p_i \cos q_i}{\varepsilon(q)}$$

yadro funktsiyalariga ega. G_o ning $\sin p_1$, $\sin p_2$, va $\sin p_3$ funktsiyalar(bazislar)ga tortilgan uch o'lchamli $S \subset C_o(\mathbb{T}^3)$ invariant qism fazodagi qismiga mos matritsa $G_o|_S$

$$G_o|_S = \begin{pmatrix} \lambda s & 0 & 0 \\ 0 & \lambda s & 0 \\ 0 & 0 & \lambda s \end{pmatrix} \tag{6.9}$$

ko'rinishdagi dioganal matritsadir. G_e operatorning 1, $\cos p_1$, $\cos p_2$ va $\cos p_3$ funktsiyalar(bazislar)ga tortilgan to'rt o'lchamli invariant qism fazodagi qismiga mos matritsa $G_e|_C$

$$G_e|_C = \begin{pmatrix} \lambda b & \lambda d & \lambda d & \lambda c \\ \lambda d & \lambda b & \lambda d & \lambda c \\ \lambda d & \lambda d & \lambda b & \lambda c \\ \mu c & \mu c & \mu c & \mu a \end{pmatrix} \tag{6.10}$$

kabi beriladi.
(6.10) dan kelib chiqadiki, agar λ, μ va γ lar

$$\lambda(b + 2d + c\gamma) = -1, \tag{6.11}$$

$$\mu(3c + a\gamma) = -\gamma \tag{6.12}$$

tengliklarni qanoatlantirsa,
u holda

$$\begin{pmatrix} \lambda b & \lambda d & \lambda d & \lambda c \\ \lambda d & \lambda b & \lambda d & \lambda c \\ \lambda d & \lambda d & \lambda b & \lambda c \\ \mu c & \mu c & \mu c & \mu a \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \gamma \end{pmatrix}.$$

Berilgan $\lambda \in \mathbb{R}$, $\lambda \neq -\frac{2a}{c}$ uchun (6.2) va (6.3) tengliklardan foydalanib (6.11) va (6.12) tenglamalarni μ va γ larga nisbatan yechsak,

$$\gamma(\lambda) = -\frac{1}{\lambda c} - 3, \tag{6.13}$$

$$\mu(\lambda) = -\frac{\gamma(\lambda)}{3c + a\gamma(\lambda)} = \frac{1+3\lambda c}{3\lambda c^2 - 3\lambda ac - a} = -\frac{1+3\lambda c}{a + \frac{\lambda c}{2}} \tag{6.14}$$

ga ega bo'lamiz. Shuningdek, agar $\lambda \neq -\frac{2a}{c}$ va $\mu = \mu(\lambda)$ (6.14) ni qanoatlantirsa, $G(0)$ operator $\psi(p) = \gamma(\lambda) + \sum_{i=1}^{3}\cos p_i$ xos funktsiyaga ega va bundan tashqari (6.13) dan

$$\psi(0) = \left(\gamma(\lambda) + \sum_{i=1}^{3}\cos p_i\right)\Big|_{p_1=p_2=p_3=0} \neq 0$$

ekanligi kelib chiqadi.

Demak, $h_{\lambda,\mu(\lambda)}$ Hamiltonian virtual sathga ega.

(6.9) matritsa tasviridan agar

$$\lambda s = -1$$

bo'lsa, ixtiyoriy $\mu \in \mathbb{R}$ uchun G_o operator uch karrali -1 xos qiymatgava unga mos $\sin p_i,\ i = 1, 2, 3$ funktsiyalarga tortilgan uch o'lchamli xos qism fazoga ega ekanligi kelib chiqadi. Xususan, $G(0)\varphi = -\varphi$ bunda $\varphi(p) = \sin p_1$ va demak, $\varphi(0) = 0$.

Xuddi shuningdek, agar

$$\lambda(b - d) = -1$$

((6.10) bilan taqqoslang)

bo'lsa, u holda ixtiyoriy $\mu \in \mathbb{R}$ uchun $G_e|_C$ operator ikki karrali -1 xos qiymatga va unga mos ikkita chiziqli bog'lanmagan $\cos p_1 - \cos p_2$ va $\cos p_1 - \cos p_3$ xos funktsiyalarga ega.

Xususan, $G\varphi = -\varphi$ bunda $\varphi(p) = \cos p_1 - \cos p_2$ va demak, $\varphi(0) = 0$.

Shunday qilib, agar $\lambda = -\frac{1}{s}$ yoki $\lambda = -\frac{1}{b-d}$ bo'lsa, u holda ixtiyoriy $\mu \in \mathbb{R}$ uchun $h_{\lambda,\mu}$ operator uning (absolyut) uzluksiz spektri tubida xos qiymatga ega bo'ladi.

$-\lambda \in \Lambda = \left\{\frac{1}{s}, \frac{1}{b-d}\right\} \setminus \left\{\frac{2a}{c}\right\}$ (6.3 natijaga ko'ra bo'sh emas) va $\mu = \mu(\lambda) = -\frac{1+3\lambda c}{a+\frac{\lambda c}{2}}$ deb olsak, $h_{\lambda,\mu(\lambda)}$ Hamiltonian uchun virtual sath va bo'sag'a xos qiymatlarining bir vaqtning o'zida mavjudligi isbotlanadi.

$\square$

Eslatma 6.5. *Biz Λ to'plamning bir yoki ikki elementdan iborat ekanligini aniqlay olmadik va shuning uchun nol energiyali xos qiymatning necha karrali ekanligi ayta olmaymiz (a, b, c, va d integrallarning sonli qiymatlari haqida ko'proq ma'lumot zarur edi). Agar Λ ikkita elementni saqlaganda edi (ikkita elementdan iborat bo'lganda edi),u holda $\hat{h}_{\lambda,-\frac{1+3\lambda c}{a+\frac{\lambda c}{2}}}$ Hamiltonian virtual sathga va $-\lambda \in \Lambda$ ning tanlanishiga bog'liq holda karrasi ikki yoki uch bo'lgan ($\lambda = -\frac{1}{b-d}$ yoki $\lambda = -\frac{1}{s}$ mos holda) bo'sag'a xos qiymatga ega.*

Agar $|\Lambda| = 1$ bo'lsa, $\hat{h}_{\lambda,-\frac{1+3\lambda c}{a+\frac{\lambda c}{2}}}$, $-\lambda \in \Lambda$ Hamiltonian virtual sathga va karralisi mos ravishda quyidagi

(i) $\frac{c}{2a} = s \neq b - d$;

(ii) $\frac{c}{2a} = b - d \neq s$;

(iii) $\frac{c}{2a} \neq s = b - d$;

hollarni o'rinli bo'lishiga bog'liq holda ikki uch hattoki besh karrali bo'sag'a xos qiymatga ega.

7. Xos qiymatlarning chekliligi

7.1. **Asosiy farazlar (tasdiqlar va shartlar).** Mazkur ish davomida biz quyidagi texnik shartlarni o'rinli deb faraz qilamiz.

Talab 7.1. $\hat{\varepsilon}_\alpha(s),\ \alpha = 1, 2$ *funktsiyalar quyidagi shartlarni qanoatlantirsin:*

(i)

$\hat{\varepsilon}_\alpha(s)$ *faqat* $s = (s^{(1)}, s^{(2)}, s^{(3)}) \in \mathbb{Z}^3$ *ning moduli* $|s| = |s^{(1)}| + |s^{(2)}| + |s^{(3)}|$ *ga bog'liq ,*

(ii)

Shunday $a, C > 0$ *sonlar mavjudki,* $|\hat{\varepsilon}_\alpha(s)| \le C \exp(-a|s|),\ s \in \mathbb{Z}^3$ *tengsizlik o'rinli,*

(iii)

$\hat{\varepsilon}_\alpha(s) < 0,\ |s| = 1$ *va* $\hat{\varepsilon}_\alpha(s) \le 0,\ |s| > 1,\ s \in \mathbb{Z}^3$.

Eslatma 7.2.

$$m_\alpha = 3(-\sum_{s\in\mathbb{Z}^3} [(s^{(1)})^2 + (s^{(2)})^2 + (s^{(3)})^2]\hat{\varepsilon}_\alpha(s))^{-1} > 0$$

son α zarrachaning effektiv massasi deb talqin qilinadi.

Talab 7.3. *Faraz qilaylik $\hat{v}(s)$ funktsiya $\mathbb{Z}^3$ da aniqlangan haqiqiy qiymatli, nomanfiy , juft va*

$$\lim_{|s|\to\infty} |s|^{3+\kappa}\hat{v}(s) = 0, \quad \kappa > 0$$

tenglikni qanoatlantirsin.

Teorema 7.4. *Faraz qilaylik 7.1 va 7.3 talablar bajarilsin. U holda ixtiyoriy $k \in U_\delta(0),\ \delta > 0$ uchun, $h(k)$ operatorning $\sigma_{ess}(h(k))$ muhim spektrdan tashqaridagi xos qiymatlari chekli.*

Lemma 7.5. *Faraz qilaylik, 7.1 talab o'rinli bo'lsin.*

U holda $\mathbb{T}^3$ da $\varepsilon_\alpha(p), \alpha = 1, 2$ juft, haqiqiy analitik funktsiya bo'lib, $p = 0$ nuqtada yagona aynimagan minimumga ega.

Proof. 7.1 talabning (i) va (ii) bandlari hamda Fourier almashtirishining xossalaridan $\varepsilon_\alpha(p)$ funktsiyaning juft va $\mathbb{T}^3$ da haqiqiy analitik bo'lishi kelib chiqadi.

7.1 talabning (i) bandiga ko'ra $\varepsilon_\alpha(p)$ funktsiya uchun

$$\varepsilon_\alpha(p) = \sum_{s\in\mathbb{Z}^3} \hat{\varepsilon}_\alpha(s)e^{i(p,s)} = \sum_{s\in\mathbb{Z}^3} \hat{\varepsilon}_\alpha(s)\cos(s^{(1)}p^{(1)})\cos(s^{(2)}p^{(2)})\cos(s^{(3)}p^{(3)}) \tag{7.1}$$

yoyilma o'rinli bo'ladi.

Shuning uchun $\varepsilon_\alpha(p)$ funktsiyaning ikkinchi tartibli hosilalari

$$\frac{\partial^2\varepsilon_\alpha}{\partial p^{(i)}\partial p^{(j)}}(0) = 0,\ i\neq j, \quad \frac{\partial^2\varepsilon_\alpha}{\partial p^{(i)}\partial p^{(i)}}(0) = \frac{1}{m_\alpha}, \quad i,j=1,2,3, \tag{7.2}$$

tenglamani qanoatlantiradi, bunda $m_\alpha > 0$ 7.2 eslatmada aniqlangan son .

Shunday qilib $\varepsilon_\alpha(p)$ funktsiyani $p=0$ da Teylor qatoriga yoysak

$$\varepsilon_\alpha(p) = \varepsilon_\alpha(0) + \frac{p^2}{2m_\alpha} + \tilde{\varepsilon}_\alpha(p), \quad \text{bunda } \tilde{\varepsilon}_\alpha(p) = O(|p|^4) \qquad p\to 0 \tag{7.3}$$

ni hosil qilamiz.

(7.3) tenglik $p=0$ nuqta $\varepsilon_\alpha(p)$ ning aynimagan minimum nuqtasi bo'lishini ko'rsatadi.

Shuning uchun (7.1) ga asosan

$$\varepsilon_\alpha(p) - \varepsilon_\alpha(0) = -\sum_{s\in\mathbb{Z}^3} \hat{\varepsilon}_\alpha(s)[1-\cos(s^{(1)}p^{(1)})\cos(s^{(2)}p^{(2)})\cos(s^{(3)}p^{(3)})]. \tag{7.4}$$

ga ega bo'lamiz.

7.1 talabning (iii) bandi va (7.4) dan $p=0$ nuqta $\varepsilon_\alpha(p)$ ning $\mathbb{T}^3$ dagi yagona aynimagan minimumi ekanligi kelib chiqadi. □

Lemma 7.6. *$p=0$ nuqtaning shunday $U_\delta(0)$ -atrofi va $U_\delta(0)$ da aniqlangan $p(k)$ analitik funktsiya mavjudki, ixtiyoriy $k\in U_\delta(0)$ uchun $p(k)$ nuqta $\mathcal{E}_k(p)$ funktsiyaning yagona aynimagan minimumi bo'ladi.*

Proof. $\varepsilon_\alpha(p), \alpha=1,2,3$ funktsiya $p=0$ da yagona aynimagan minimumga ega bo'lganligi uchun $\bigtriangledown\varepsilon_\alpha(p)$ gradient $p=0$ nuqtada nolga teng, ya'ni

$$\bigtriangledown\varepsilon_\alpha(p)|_{p=0} = (\frac{\partial\varepsilon_\alpha(p)}{\partial p^{(1)}}, \frac{\partial\varepsilon_\alpha(p)}{\partial p^{(2)}}, \frac{\partial\varepsilon_\alpha(p)}{\partial p^{(3)}})|_{p=0} = 0.$$

Shuning uchun (8.5) ga ko'ra

$$B_\alpha(p)|_{p=0} = (\frac{\partial^2\varepsilon_\alpha(0)}{\partial p^{(i)}\partial p^{(j)}})|_{i,j=1,2,3} = m_\alpha^{-1}I_3,\ \alpha=1,2,$$

matrisa musbat bo'ladi, bunda I_3 - 3×3 o'lchamli birlik matrisa.

Bundan esa o'z navbatida $\bigtriangledown E_0^{(\alpha)}(0) = 0$ ekanligi va

$$B(0) = (\frac{\partial^2\mathcal{E}_0(0)}{\partial p^{(i)}\partial p^{(j)}})|_{i,j=1}^3 = (m_1^{-1}+m_2^{-1})I_3$$

matrisa musbat aniqlanganligi kelib chiqadi. Endi

$$\bigtriangledown\mathcal{E}_k(p) = 0,\ k,p\in\mathbb{T}^3$$

tenglamaga oshkormas funktsiya haqidagi teoremani qo'llaymiz.

U holda $k=0$ nuqtaning shunday $U_\delta(0)$-atrofi va $U_\delta(0)$ da aniqlangan analitik $p(k)$ vektor funktsiya mavjudki, barcha $k\in U_\delta(0)$ uchun $\bigtriangledown\mathcal{E}_k(p(k)) = 0$ ayniyat o'rinli bo'ladi.

$B(k)$ orqali $\mathcal{E}_k(p)$ funktsiyaning $p(k)$ nuqtadagi ikkinchi tartibli xususiy hosilalaridan tuzilgan matrisani belgilaymiz. $B(0)$ matrisa musbat hamda $U_\delta(0)$ da uzluksiz va shuninhg uchun ixtiyoriy $k\in U_\delta(0)$ da $B(k)$ matrisa musbat aniqlangan bo'ladi. Bundan $p(k),\ k\in U_\delta(0)$ nuqta $\mathcal{E}_k(p)$ funktsiyaning yagona aynimagan minimum nuqtasi bo'lishi kelib chiqadi. □

Biz ikki zarrachali diskret Shroedinger operatorlari uchun Birman-Schwinger prinsipini([55] ga qarang) keltirib chiqaramiz.

$N(k,z)$ orqali $h(k),\ k\in\mathbb{T}^3$ operatorning $z\le\mathcal{E}_{\min}(k)$ dan quyidagi xos qiymatlar sonini belgilaylik.

O'z-o'ziga qo'shma chegaralangan A operator uchun $n(\lambda, A)$ orqali

$$n(\lambda, A) = sup\{dimF : (Au,u) > \lambda,\ u\in F,\ ||u|| = 1\} \tag{7.5}$$

ni belgilaymiz. Agar $n(\lambda, A)$ cheksizga teng bo'lsa, λ son A operatorning muhim spektriga tegishli va agar $n(\lambda, A)$ ning qiymati chekli bo'lsa, u A ning λ dan katta xos qiymatlar soniga teng bo'ladi.

$N(k, z)$ ning aniqlanishidan $N(k, z) = (-z, -h(k))$ tenglik o'rinli.

Berilgan $\hat{v}$ operator $\hat{v}(s)$ musbat funktsiyaga ko'paytirish operatori bo'lganligi uchun uning musbat ildizi $\hat{v}^{\frac{1}{2}}$ ushbu $v^{\frac{1}{2}}(s)$ funktsiyaga ko'paytirish operatoridir.

Shuning uchun v operator ham musbat va v ning $v^{\frac{1}{2}}$ musbat ildizi

$$(v^{\frac{1}{2}} f)(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{T}^3} v^{\frac{1}{2}}(p - p') f(p') dp',$$

kabi aniqlanadi, bundda $v^{\frac{1}{2}}(p)$ yadro funktsiyasi $\hat{v}^{\frac{1}{2}}(s)$ funktsiyaning teskari Fourier almashtirishidan iborat, ya'ni

$$v^{\frac{1}{2}}(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{s \in \mathbb{T}^3} \hat{v}^{\frac{1}{2}}(s) e^{i(p,s)}.$$

Ixtiyoriy $k \in U_\delta(0)$, $z \le \mathcal{E}_{\min}(k)$ uchun $G(k, z)$ va $\widetilde{G}(k, z)$ integral opertorlarni

$$G(k, z; p, q) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{v(p - q)}{\sqrt{\mathcal{E}_k(p) - z}\sqrt{\mathcal{E}_k(q) - z}}$$

va

$$\widetilde{G}(k, z; p, q) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{v^{\frac{1}{2}}(p - q)}{(\mathcal{E}_k(q) - z)^{\frac{1}{2}}}.$$

yadrolar yordamida aniqlaymiz.

Ikki zarrachali diskret Shroedinger operatori uchun Birman-Schwinger prinsipining quyidagi variantini xuddi $\mathbb{R}^3$ dagi kvant zarrachalar holidagi kabi isbotlash mumkin ([55] ga qarang).

Teorema 7.7. *Hilbert fazosi $L_2(\mathbb{T}^3)$dagi $G(k, z)$, $k \in \mathbb{T}^3$, $z < \mathcal{E}_{min}(k)$ operator uchun*

$$N(k, z) = n(1, G(k, z))$$

tenglik o'rinli.

$\mathbb{C}$ kompleks sonlar maydoni bo'lsin. Ixtiyoriy $k \in \mathbb{T}^3$ uchun $\Delta_\alpha(k, z)$ orqali

$$I - v^{\frac{1}{2}} r^0(k, z) v^{\frac{1}{2}}, \quad z \in \mathbb{C} \setminus [\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)],$$

operatorning Fredgolm determinantini belgilaymiz, bunda I-$L_2(\mathbb{T}^3)$ dagi birlik operator.

$\Delta(k, z)$ funktsiya $\mathbb{T}^3 \times (\mathbb{C} \setminus \sigma_{\text{ess}}(h(k)))$ da haqiqiy analitikligini ta'kidlab o'tamiz.

Lemma 7.8. *Ixtiyoriy $k \in \mathbb{T}^3$ uchun $z \in \mathbb{C} \setminus [\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)]$ soni $h(k)$ operatorning xos qiymati bo'lishi uchun*

$$\Delta(k, z) = 0$$

bo'lishi zarur va yetarli.

Proof. Birman-Schwinger prinsipiga ko'ra (Teorema 4.3)
$z \in \mathbb{C} \setminus [\mathcal{E}_{\min}(k), \mathcal{E}_{\max}(k)]$ soni faqat va faqat

$$g = v^{\frac{1}{2}} r^0(k, z) v^{\frac{1}{2}} g \tag{7.6}$$

tenglama $\hat{g} \in L_2(\mathbb{T}^3)$ notirivial echimga ega bo'lgandagina $h(k)$, $k \in \mathbb{T}^3$ operatorning xos qiymati bo'ladi.

Fredgolm teoremasiga ko'ra (7.6) tenglama faqat va faqat

$$\Delta(k, z) = 0$$

□

bo'lgandagina no'lmas yechimga ega.

Quyidagi teorema $\mathbb{Z}^3$ panjaradagi ikki zarrachali Shroedinger operatori uchun Birman Shvinger prinsipining umumlashmasidan iborat.

Teorema 7.9. *$G(k, z)$, $k \in U_\delta(0)$ $z \le \mathcal{E}_{min}(k)$ operator $L_2(\mathbb{T}^3)$ da aniqlangan (ta'sirlashuvchi) musbat Σ_1 iz sinfga tegishli, z bo'yicha $z = \mathcal{E}_{min}(k)$ gacha chapdan uzluksiz hamda*

$$N(k, \mathcal{E}_{min}(k)) = n(1, G(k, \mathcal{E}_{min}(k))),$$

tenglik o'rinli.

Proof. 7.3 talabga ko'ra $v(p)$ funktsiya $\mathbb{T}^3$ da uzluksiz va uni

$$v(p-p') = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} v^{\frac{1}{2}}(p-t)v^{\frac{1}{2}}(p'-t)dt$$

ko'rinishda tasvirlash mumkin. Ixtiyoriy $k \in U_\delta(0)$ uchun $\mathcal{E}_k(p)$ funktsiya $p = p(k)$ nuqtada yagona aynimagan minimumga ega bo'lganligi uchun $\widetilde{G}(k, \mathcal{E}_{\min}(k); p, q)$ gunksiya $(\mathbb{T}^3)^2$ da kvadrati bilan integrallanuvchi,ya'ni $\widetilde{G}(k, \mathcal{E}_{\min}(k))$ operator Σ_2 Gilbert-Shmidt sinfiga qarashli bo'ladi. Lebegning yaqinlashish haqidagi teoremasidan $\widetilde{G}(k,z)$ ning operator qiymatli funktsiya sifatida $\mathcal{E}_{\min}(k)$ gacha chapdan uzluksizligi kelib chiqadi.

U holda

$$G(k,z) = \widetilde{G}^*(k,z)\widetilde{G}(k,z), \quad k \in U_\delta(0), \quad z \le \mathcal{E}_{\min}(k) \tag{7.7}$$

tenglikka ko'ra $G(k,z)$ operator $\mathcal{E}_{\min}(k)$ gacha chapdan uzluksiz va Σ_1 iz sinfiga qarashli.

$$(G(k,z)f, f) = ||\widetilde{G}(k,z)f||^2, \quad f \in L_2(\mathbb{T}^3)$$

tenglik $G(k,z)$ ning musbatligini isbotlaydi.

$$N(k, \mathcal{E}_{\min}(k)) = n(1, G(k, \mathcal{E}_{\min}(k)))$$

tenglikni isbotlaymiz.

$G(k, \mathcal{E}_{\min}(k))$ kompakt operator bo'lgani uchun $n(1-\eta, G(k, \mathcal{E}_{\min}(k)))$ soni ixtiyoriy $\eta < 1$ uchun chekli.

Veyl tengsizligi

$$n(\lambda_1 + \lambda_2, A_1 + A_2) \le n(\lambda_1, A_1) + n(\lambda_2, A_2)$$

ga asosan ixtiyoriy $\eta \in (0,1)$ va barcha $z < \mathcal{E}_{\min}(k)$ lar uchun

$$N(k,z) = n(1, G(k,z)) \le n(1-\eta, G(k, \mathcal{E}_{\min}(k))) + n(\eta, G(k,z) - G(k, \mathcal{E}_{\min}(k)))$$

tengsizlik o'rinli. Shuning uchun $G(k,z)-$ $\mathcal{E}_{\min}(k)$ gacha chapdan uzluksiz bo'lganidan,

$$\lim_{z \to \mathcal{E}_{\min}(k)-0} N(k,z) \le n(1-\eta, G(k, \mathcal{E}_{\min}(k))), \eta \in (0,1)$$

tengsizlikni hosil qilamiz. Shunday qilib,

$$\lim_{z \to \mathcal{E}_{\min}(k)-0} N(k,z) = N(k, \mathcal{E}_{\min}(k)) \tag{7.8}$$

tenglikdan

$$N(k, \mathcal{E}_{min}(k)) \le \lim_{\varepsilon \to 0} n(1-\varepsilon, G(k, \mathcal{E}_{min}(k))) < \infty$$

ekanligi kelib chiqadi.

F.q. $f \in \mathcal{H}_{-h(k)}\big(-(\mathcal{E}_{\min}(k) - \varepsilon)\big)$ bo'lsin. U holda

$$((h^0_\alpha(k) - \mathcal{E}_{\min}(k))f, f) < ((h^0(k) - \mathcal{E}_{\min}(k) + \varepsilon)f, f) < (vf, f)$$

tengsizlik o'rinli.

$y = (h^0(k) - \mathcal{E}_{\min}(k))^{\frac{1}{2}} f$ deb olsak

$$(y,y) < (\widetilde{G}^*(k, \mathcal{E}_{\min}(k))\widetilde{G}(k, \mathcal{E}_{\min}(k))y, y)$$

ga ega bo'lamiz.

Shunday qilib,

$$N(k, \mathcal{E}_{\min}(k)) \le n(1, G(k, \mathcal{E}_{min}(k))). \tag{7.9}$$

F.q. $\varrho > 0$ etarlicha kichik son va $\varphi \in \mathcal{H}_{G(k, \mathcal{E}_{\min}(k))}\big(1+\varrho\big)$ bo'lsin. $G(k,z)$ operator $z = \mathcal{E}_{\min}(k)$, $k \in U_\delta(0)$ gacha chapdan uzluksiz bo'lgani uchun shunday $\xi > 0$ mavjudki, barcha $z \in (\mathcal{E}_{\min}(k) - \xi, \mathcal{E}_{\min}(k))$ lar uchun

$$||G(k,z) - G(k, \mathcal{E}_{\min}(k))|| < \varrho$$

tengsizlik o'rinli.

Bundan

$$\begin{aligned}(G(k,z)\varphi, \varphi) &= (G(k, \mathcal{E}_{\min}(k))\varphi, \varphi) \\ &\quad + ((G(k,z) - G(k, \mathcal{E}_{\min}(k)))\varphi, \varphi) > (\varphi, \varphi)\end{aligned}$$

ni hosil qilamiz.

Demak,$k \in U_\delta(0),\ z \in (\mathcal{E}_{\min}(k) - \xi, \mathcal{E}_{\min}(k))$ va etarlicha kichik $\varrho > 0$ uchun

$$n(1, G(k,z)) \geq n(1+\varrho, G(k, \mathcal{E}_{\min}(k))).$$

tengsizlikka egamiz.

Birman- Schwinger prinsipiga (Theorem 7.7) va (7.8) tenglikka asosan

$$N(k, \mathcal{E}_{min}(k)) \geq n(1+\varrho, G(k, \mathcal{E}_{\min}(k))) \tag{7.10}$$

ni hosil qilamiz.

Ixtiyoriy $k \in U_\delta(0)$ uchun $n(\lambda, G(k, \mathcal{E}_{\min}(k))), \lambda > 0$ funktsiya o'ngdan uzluksiz bo'lgani uchun, (7.10) tengsizlik (7.9) bilan birgalikda Teorema 12.6 ning isbotini yakunlaydi. □

Teorema 7.4 ning isboti. v operator musbat, shuning uchun $h(k)$ operatorning $\sigma_{ess}(h(k))$ dan o'ngda yotuvchi xos qiymati yo'q. $h(k)$ operatorning $\mathcal{E}_{\min}(k))$ dan quyida joylashgan $\sigma_d(h(k))$ diskret spektrning chekliligi $G(k, \mathcal{E}_{\min}(k))$ ning kompaktligi va (7.9) tengsizlikdan kelib chiqadi.

□

Quyidagi misol ikki zarrachali $h(k), k \in \mathbb{T}^3$ operatorning $\sigma_{ess}(h(k))$ muhim spektrdan quyida joylashgan cheksiz ko'p xos qiymatga ega bo'lishi mumkinligini ko'rsatadi.

Misol 7.10.

$$\hat{\varepsilon}_\alpha(s) = \begin{cases} 3, & s = 0 \\ -\frac{1}{2}, & |s| = 1 \\ 0, & \textit{boshqa hollarda} \end{cases}$$

va

$$\hat{v}(s) = \begin{cases} e^{-|s^{(3)}|}, & s = (s^{(1)}, s^{(2)}, s^{(3)}) \in \mathbb{Z}^3,\ s^{(1)} = s^{(2)} = 0 \\ 0, & \textit{boshqa hollarda}. \end{cases}$$

bo'lsin.

$\hat{\varepsilon}_\alpha(s)$ va $\hat{v}(s)$ funktsiyalar mos ravishda bu funktsiyalarga qo'yilgan 1) va 2) talablarni qanoatlantiradi.

Bu $\varepsilon_\alpha(p)$ va $v(p)$ funktsiyalarning $\hat{\varepsilon}_\alpha(s)$ va $\hat{v}(s)$ Fur'e almashtirishlari mos ravishda

$$\varepsilon_\alpha(p) = \sum_{i=1}^{3} (1 - \cos p^{(i)})$$

va

$$v(p) := v(p^{(3)}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{n \in \mathbb{Z}^1} e^{-|n|} \cos np^{(3)}.$$

ko'rinishda bo'ladi.

$h^0(k),\ k = (k^{(1)}, k^{(2)}, \pi) \in \mathbb{T}^3$ operator $L_2(\mathbb{T}^3)$ da

$$\mathcal{E}_k(p) = \mathcal{E}_k(p^{(1)}, p^{(2)}) = 2 + \sum_{i=1}^{2} (2 - \cos p^{(i)} - \cos(k^{(i)} - p^{(i)}))$$

funktsiyaga ko'paytirish operatori va v $L_2(\mathbb{T}^3)$ da

$$(v\psi)(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{T}^3} v(p^{(3)} - t^{(3)}) \psi(t) dt$$

ko'rinishdagi integral operator bo'ladi.

Osongina ko'rish mumkinki, $\{e^{-|n|},\ n \in \mathbb{Z}^1\}$ sonlar $\hat{v}$ operator uchun xos qiymatlar bo'ladi. Shuningdek ular v operatorning ham xos qiymati bo'ladi, ya'ni

$$(v\psi_n)(p) = (2\pi)^{\frac{1}{2}} \int_{\mathbb{T}^1} v(p^{(3)} - t^{(3)}) \psi_n(t^{(3)}) dt^{(3)} = \hat{v}(n) \psi_n(p^{(3)}) \tag{7.11}$$

va

$$\psi_n(p) := \psi_n(p^{(3)}) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{\mathrm{i} p^{(3)} n}$$

lar shu xos qiymatga mos xos vektorlar bo'ladi.

$$\lim_{z \to \mathcal{E}_{\min}(k) - 0} \int_{\mathbb{T}^2} \frac{dp^{(1)} dp^{(2)}}{\mathcal{E}_k(p^{(1)}, p^{(2)}) - z} = +\infty$$

bo'lgani uchun ixtiyoriy $n \in \mathbb{Z}^1$ uchun

$$1 - \frac{\hat{v}(n)}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{dp}{\mathcal{E}_k(p^{(1)}, p^{(2)}) - z_n} = 0$$

tenglama $n \to \infty$ da $z_n \to \mathcal{E}_{\min}(k)$ ni qanoatlantiruvchi $z_n < \mathcal{E}_{\min}(k)$ yechimga ega bo'ladi.

Quyidagicha

$$f_n(p) = \frac{1}{\mathcal{E}_k(p^{(1)}, p^{(2)}) - z_n} \psi_n(p^{(3)}), \quad n \in \mathbb{Z}^1$$

belgilash olamiz.

U holda , $f_n \perp f_m$, $n \neq m$, $n, m \in \mathbb{Z}^1$ ga ega bo'lamiz .

(7.11) dan foydalanib

$$(h(k) - z_n) f_n = \Big(1 - \frac{\hat{v}(n)}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{dp^{(1)} dp^{(2)}}{\mathcal{E}_k(p^{(1)}, p^{(2)}) - z_n}\Big) \psi_n = 0$$

ni hosil qilamiz.

Bundan esa f_n, $n \in \mathbb{Z}^1$ funktsiyalar z_n, $n \in \mathbb{Z}^1$ ga mos xos funktsiyalar degan xulosaga kelamiz.

8. (ZARRACHALARI "KONTAKT"(NUQTADA) TA'SIRLASHGAN SISTEMAGA MOS SCHRÖDINGER OPERATORI $h(k)$ NING SPEKTRAL XOSSALARI

Bu bo'limda biz ikki zarrachali $h(k)$, $k \in \mathbb{T}^3$ diskret Shroedinger operatorining spektral xossalarini o'rganamiz.

F.q. $h(k)$, $k \in \mathbb{T}^3 - L_2(\mathbb{T}^3)$ Gilbert fazosida quyidagicha aniqlangan o'z-o'ziga qo'shma operatorlar oilasi bo'lsin:

(8.1) $$h(k) = h^0(k) - \mu v.$$

$L_2(\mathbb{T}^3)$ dagi qo'zg'almagan $h^0(k)$ operator $\mathcal{E}_k(p)$ funktsiyaga ko'paytirish operatori

$$(h^0(k) f)(p) = \mathcal{E}_k(p) f(p), \quad f \in L_2(\mathbb{T}^3),$$

bunda $\mathcal{E}_k(p)$ funktsiya (10.7) da aniqlangan.

v qo'zg'alishi bir o'lchamli (rangi birga teng bo'lgan) integral operator

$$(vf)(p) = (2\pi)^{-3} \int\limits_{\mathbb{T}^3} f(q) dq, \quad f \in L_2(\mathbb{T}^3).$$

$$m = \frac{m_1 m_2}{m_1 + m_2}, \quad \alpha, \beta = 1, 2, \quad \alpha \neq \beta$$

belgilash olamiz.

C kompleks tekislik bo'lsin. Ixtiyoriy $k \in \mathbb{T}^3$ va $z \in \mathbf{C} \setminus \sigma_{\text{cont}}(h(k))$ uchun

$$\Delta(k, z) = 1 - \mu (2\pi)^{-3} \int\limits_{\mathbb{T}^3} (\mathcal{E}_k(q) - z)^{-1} dq.$$

funktsiyani (($h(k)$) operatorga mos Fredgolm determinantini) aniqlaymiz.

$\Delta(k, z)$ funktsiya $\mathbb{T}^3 \times (\mathbf{C} \setminus \sigma_{\text{ess}}(h(k)))$ da haqiqiy analitikligini ta'kidlab o'tamiz.

Quyidagi lemma Birman-Schwinger prinsipi va Fredgol'm teoremasining sodda natijasidir.

Lemma 8.1. *$k \in \mathbb{T}^3$ bo'lsin. $z \in \mathbf{C} \setminus \sigma_{ess}(h(k))$ soni $h(k)$ opreratorning xos qiymati bo'lishi uchun*

$$\Delta(k, z) = 0$$

bo'lishi zarur va yetarli. □

Lemma 8.2. *Quyidagi tasdiqlar ekvivalent :*
(i) $h(0)$ operator nol energiyali rezonansiga ega.
(ii) $\Delta(0, 0) = 0$;
(iii) $\mu = \mu^0$.

Proof. Biror $\mu > 0$ uchun $h(0)$ operator nol energiya rezonansiga ega bo'lsin. U holda Ta'rif 12.3 ga ko'ra

$$\varphi(p) = \mu m (2\pi)^{-3} \int\limits_{\mathbb{T}^3} (\varepsilon(q))^{-1} \varphi(q) dq$$

tenglama $C(\mathbb{T}^3)$ da oddiy echimga ega va $\varphi(q)$ yechim 1 ga teng.

Bundan ko'rinadiki

$$1 = \mu m(2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq$$

va shuning uchun

$$\Delta(0,0) = 1 - \mu m(2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq = 0$$

va demak, $\mu = \mu^0$.

Biror $\mu > 0$ uchun

$$\Delta(0,0) = 1 - \mu m(2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq = 0$$

tenglik o'rinli bo'lsin va o'z navbatida $\mu = \mu^0$ bo'lsin. U holda faqatgina $\varphi(q) \equiv constant \in C(\mathbb{T}^3)$ funktsiya

$$\varphi(p) = \mu m(2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} \varphi(q) dq,$$

tenglamaning echimi bo'ladi, ya'ni $h(0)$ operator nol energiyali rezonansga ega bo'ladi. □

Teorema 8.3. *$h(0)$ operator nol energiyali rezonansga ega bo'lsin. U holda barcha $k \in \mathbb{T}^3$, $k \neq 0$ lar uchun $h(k)$ operator muhim spektrdan quyida yagona oddiy $z(k)$ xos qiymatga ega bo'ladi. Bundan tashqari $z(k)$ funktsiya $\mathbb{T}^3 \setminus \{0\}$ da juft va $k \neq 0$ bo'lganda $z(k) > 0$ bajariladi.*

Proof. Lemma 15.2 ga ko'ra

$$\Delta(0,0) = 1 - \mu^0 m(2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq = 0$$

va shuning uchun osongina ko'rish mumkinki, ixtiyoriy $z < 0$ uchun $\Delta(0,z) > 0$ tengsizlik o'rinli. Lemma 11.3 ga ko'ra $h(0)$ operator manfiy xos qiymatlarga ega emas. $p = p(k)$ nuqta $\mathcal{E}_k(p)$ funktsiyaning aynimagan minimumi bo'lganligi uchun $\Delta(k, \mathcal{E}_{\min}(k))$ ni

$$\Delta(k, \mathcal{E}_{\min}(k)) = 1 - \mu^0 (2\pi)^{-3} \int_{\mathbb{T}^3} (E_k^{(\alpha)}(q) - \mathcal{E}_{\min}(k))^{-1} dq$$

kabi aniqlaymiz.

Lebegning yaqinlashishi haqidagi teoremasiga ko'ra

$$\lim_{z \to \mathcal{E}_{\min}(k)} \Delta(k, z) = \Delta(k, \mathcal{E}_{\min}(k))$$

munosabatga egamiz.

Barcha $k \neq 0, q \neq 0$ uchun

$$\mathcal{E}_k(q + p(k)) - \mathcal{E}_{\min}(k) < E_0(q)$$

tengsizlik o'rinli va shuning uchun quyidagi

$$\Delta(k, \mathcal{E}_{\min}(k)) < \Delta(0,0) = 0, k \neq 0 \tag{8.2}$$

tengsizlikni hosil qilamiz.

Har bir $k \in \mathbb{T}^3$ uchun $\Delta(k, \cdot)$ funktsiya $(-\infty, \mathcal{E}_{\min}(k)]$ da monoton kamayuvchi va $z \to -\infty$ da $\Delta(k, z) \to 1$. U holda (8.2) ga asosan shunday $z(k) \in (-\infty, \mathcal{E}_{\min}(k))$ son mavjudki, $\Delta(k, z(k)) = 0$ bo'ladi. Lemma 11.3 da ko'ra ixtiyoriy nolmas $k \in \mathbb{T}^3$ uchun $h(k)$ operator $\mathcal{E}_{\min}(k)$ dan quyida xos qiymatga ega. Ixtiyoriy $k \in \mathbb{T}^3$ va $z \in (-\infty, \mathcal{E}_{\min}(k))$ uchun $\Delta(-k, z) = \Delta(k, z)$ tenglik o'rinli va shuning uchun $z(k)$ juft.

$z(k),\ k \neq 0$ xos qiymatning musbatligini isbotlaylik. Dastlab, biz barcha $k \in \mathbb{T}^3, k \neq 0$ uchun

$$\Delta(k, 0) > 0 \tag{8.3}$$

tengsizlikni ko'rsatamiz.

μ^0 ning (7.3) ko'rinishida aniqlanishini hisobga olsak,

$$\Delta(k, 0) = \mu^0 (2\pi)^{-3} \int_{\mathbb{T}^3} \frac{\varepsilon_2(k-p) - \varepsilon_2(p)}{\mathcal{E}_0(p)\mathcal{E}_k(p)} dp \tag{8.4}$$

ga ega bo'lamiz.

(8.4) da $q = \frac{k}{2} - p$ o'zgaruvchilarni almashtirib, va $\Delta(k,0) = \Delta(-k,0)$ tenglikni qo'llab,

$$\Delta(k,0) = \frac{\Delta(k,0) + \Delta(-k,0)}{2} =$$

$$= \frac{\mu^0}{2}(2\pi)^{-3} \int\limits_{\mathbb{T}^3} (\varepsilon_2(\frac{k}{2}+p) - \varepsilon_2(\frac{k}{2}-p))(\varepsilon_1(\frac{k}{2}+p) - \varepsilon_1(\frac{k}{2}-p))F(k,p)dp,$$

tenglikka kelamiz,

bunda

$$F(k,p) = \frac{\mathcal{E}_0(\frac{k}{2}+p) + \mathcal{E}_0(\frac{k}{2}-p)}{\mathcal{E}_0(\frac{k}{2}+p)\mathcal{E}_0(\frac{k}{2}-p)\mathcal{E}_k(\frac{k}{2}+p)\mathcal{E}_k(\frac{k}{2}-p)} > 0.$$

Sodda hisoblashlardan ko'rinadiki,

$$(\varepsilon_2(\frac{k}{2}+p) - \varepsilon_2(\frac{k}{2}-p))(\varepsilon_1(\frac{k}{2}+p) - \varepsilon_1(\frac{k}{2}-p)) = \frac{4}{m_1 m_2}\left(\sum_{i=1}^{3} cos\frac{k^{(i)}}{2} cosp^{(i)}\right)^2 \geq 0.$$

Shunday qilib, (8.3) tengsizlik isbotlandi.

Ixtiyoriy $k \in \mathbb{T}^3$ uchun $\Delta(k,\cdot)$ funktsiya monoton kamayuvchi va

$$\Delta(k,0) > \Delta(k,z(k)) = 0 > \Delta(k,\mathcal{E}_{\min}(k)), \quad k \neq 0$$

tengsizliklar o'rinli. Demak, $h(k)$ operatorning $z(k)$ xos qiymati $(0,\mathcal{E}_{\min}(k))$ intervalga tegishli. □

Lemma 8.4. *$\mu = \mu^0$, bo'lsin. U holda ixtiyoriy $k \in U_\delta(0)$ va $z \leq \mathcal{E}_{\min}(k)$ uchun quyidagi:*

$$\Delta(k,z) = \frac{\mu^0 m^{3/2}}{\sqrt{2}\pi}\left[\mathcal{E}_{\min}(k) - z\right]^{\frac{1}{2}} + \Delta^{(20)}(\mathcal{E}_{\min}(k) - z) + \Delta^{(02)}(k,z)$$

yoyilma o'rinli, bunda $z \to \mathcal{E}_{\min}(k)$ da $\Delta^{(20)}(\mathcal{E}_{\min}(k) - z) = O(\mathcal{E}_{\min}(k) - z)$ va $k \to 0$ da $\Delta^{(02)}(k,z) = O(|k|^2)$

Proof.

$$E(k,p) = \mathcal{E}_k(p + p(k)) - \mathcal{E}_{\min}(k)$$

bo'lsin, bunda $p(k) \in \mathbb{T}^3$ $\mathcal{E}_k(p)$ ning minimum nuqtasi, ya'ni $\mathcal{E}_{\min}(k) = \mathcal{E}_k(p(k))$. U holda (**??**) ni qo'llab,

$$E(k,p) = \sum_{j=1}^{3} r(k^{(j)})(1 - \cos p^{(j)})$$

tenglikka kelamiz.

$\mathbb{T}^3 \times \mathbf{C}_+$ da $\tilde{\Delta}(k,w)$ funktsiyani $\tilde{\Delta}(k,w) = \Delta(k,\mathcal{E}_{\min}(k) - w^2)$ kabi aniqlaymiz, bunda $\mathbf{C}_+ = \{z \in \mathbf{C} : \mathrm{Re} z > 0\}$.

$\tilde{\Delta}(k,w)$ funktsiyani

$$\tilde{\Delta}(k,w) = 1 - \mu(2\pi)^{-3} \int\limits_{\mathbb{T}^3} \frac{dp}{E(k,p) + w^2}$$

$$= 1 - \mu(2\pi)^{-3} \int_{\mathbb{T}^3} \frac{dp}{\sum_{j=1}^{3} r(k^{(j)})(1 - \cos p^{(j)}) + w^2}$$

ko'rinishida tasvirlaymiz.

$V_\delta(0) -$ $w = 0 \in \mathbf{C}$ nuqtaning kompleks δ-atrofi bo'lsin.

$\Delta^*(k,w)$ orqali $\tilde{\Delta}(k,w)$ funktsiyaning $\mathbb{T}^3 \times (\mathbf{C}_+ \cup V_\delta(0))$ sohaga analitik davomini belgilaymiz. Bu funktsiya $k \in \mathbb{T}^3$ bo'yicha juft.

Shunday qilib,

$$\Delta^*(k,w) = \Delta^*(k,w) + \tilde{\Delta}^{(20)}(k,w),$$

bunda $k \to 0$ da $\tilde{\Delta}^{(20)}(k,w) = O(|k|^2)$ tenglik $w \in \mathbf{C}_+$ bo'yicha tekis yaqinlashadi.

Taylor qatoriga ko'ra

$$\Delta^*(k,w) = \tilde{\triangle}^{(01)}(0,0)w + \tilde{\triangle}^{(02)}(0,w)w^2,$$

bunda $w \to 0$ da $\tilde{\triangle}^{(02)}(0,w) = O(1)$.

Sodda hisoblashlardan keyin

$$\frac{\partial \Delta^*(0,0)}{\partial w} = \tilde{\triangle}^{(01)}(0,0) = \frac{\mu^0 m^{3/2}}{\sqrt{2}\pi} \neq 0 \tag{8.5}$$

□

tenglikka ega bo'lamiz.

Natija 8.5. $z(k) = \mathcal{E}_{\min}(k) - w^2(k)$ *funktsiya* $U_\delta(0)$ *da haqiqiy analitik, bunda* $w(k-$ $\tilde{\Delta}(k,w) = 0$ *tenglamaning yagona oddiy yechimi va* $k \to 0$ *da* $w(k) = O(|k|^2)$.

Proof. $\tilde{\Delta}(0,0) = 0$ va (8.5) tengsizlik o'rinli bo'lganligi uchun $\tilde{\Delta}(k,w) = 0$ tenglama yagona oddiy $w(k),\ k \in U_\delta(0)$ yechimga ega va $U_\delta(0)$ da haqiqiy analitik. $\tilde{\Delta}(k,w)$ funktsiya $k \in U_\delta(0), \delta > 0$ da juft va $w(0) = 0$ ekanligini e'tiborga olib, $w(k) = O(|k|^2)$ ga ega bo'lamiz. Shunday qilib, $z(k) = \mathcal{E}_{\min}(k) - w^2(k)$ funktsiya $U_\delta(0)$ da haqiqiy analitik. □

Lemma 8.6. *F.q.* $\mu = \mu^0$ *bo'lsin. U holda ixtiyoriy* $k \in U^0_\delta(0)$ *uchun shunday* $\delta(k) > 0$ *son mavjudki, barcha* $z \in V_{\delta(k)}(z(k))$ *uchun quyidagi*

$$\Delta(k,z) = C_1(k)(z - z(k))\hat{\Delta}(k,z)$$

yoyilma o'rinli.

Bu yerda $C_1(k) \neq 0,$, $V_{\delta(k)}(z(k))-$ $z(k)$ *nuqtaning* $\delta(k)$ *atrofi va* $\hat{\Delta}(k,z)$ *funktsiya* $V_{\delta(k)}(z(k))$ *da uzluksiz va* $\hat{\Delta}(k,z(k)) \neq 0.$

Proof. $z(k) < \mathcal{E}_{\min}(k),\ k \neq 0$ bo'lganligi uchun $\Delta(k,z)$ funktsiya

$$\Delta(k,z) = \sum_{n=1}^{\infty} C_n(k)(z - z(k))^n, \quad z \in V_{\delta(k)}(z(k)),$$

kabi tasvirlanishi mumkin, bunda

$$C_1(k) = \frac{\mu^o m^{3/2}}{\sqrt{2}\pi} \frac{1}{2\sqrt{\mathcal{E}_{\min}(k) - z(k)}} \neq 0, \quad k \neq 0.$$

Shuning uchun $\hat{\Delta}(k,z)$ $V_{\delta(k)}(z(k))$da uzluksiz. $z(k),\ k \neq 0$ soni $\Delta(k,z) = 0,\ z \leq \mathbb{E}_{\min}(k)$ tenglamaning oddiy yechimi bo'lganligi uchun $\hat{\Delta}(k, z_(k)) \neq 0$ ga ega bo'lamiz. □

9. Panjaradagi ikkita bir xil zarrachalar sistemasi Hamiltonianlari haqida

10. Panjaradagi ikkita bir xil zarrachalar sistemasi Hamiltonianining koordinata va impuls tasvirlari(ko'rinishlari)

$\mathbb{Z}^d-$ d o'lchamli panjara va $(\mathbb{Z}^d)^2 - \mathbb{Z}^d$ ning 2-darajali Dekart ko'paytmasi bo'lsin. $\ell_2((\mathbb{Z}^d)^2)$ orqali $(\mathbb{Z}^d)^2$ da aniqlangan kvadrati bilan jamlanuvchi φ funktsiyalar fazosini va $\ell^s_2((\mathbb{Z}^d)^2) \subset \ell_2((\mathbb{Z}^d)^2)$ orqali simmetrik funktsiyalar qism fazosini belgilaymiz.

Ikkita bir xil zarracha erkin Hamiltoniani $\widehat{h}^0$ ikkita bir xil $\hat{\varepsilon}_\alpha(\cdot) = \hat{\varepsilon}(\cdot)$ funsiyalar orqali quyidagicha aniqlanadi

$$(\hat{h}^0\hat{\psi})(x_1,x_2) = \sum_{s\in\mathbb{Z}^d} \hat{\varepsilon}(s)[\hat{\psi}(x_1+s,x_2) + \hat{\psi}(x_1,x_2+s)], \quad \hat{\psi} \in \ell^s_2((\mathbb{Z}^d)^2)$$

O'zaro ta'sirlashuvchi ikkita bir xil kvant zarrachalar sistemasi to'la $\hat{h}$ Hamiltoniani $\ell^{(s)}_2((\mathbb{Z}^d)^2)$ Gilbert fazosida o'z-o'ziga qo'shma operatorni aniqlaydi va quyidagi ko'rinishga ega bo'ladi:

$$\widehat{h} = \widehat{h}^0 - \widehat{v}, \tag{10.1}$$

bunda

$$(\widehat{v}\hat{\psi})(x_\beta,x_\gamma) = \hat{v}(x_\beta - x_\gamma)\hat{\psi}(x_\beta,x_\gamma), \beta \neq \gamma,\ \beta,\gamma = 1,2,3, \quad \hat{\psi} \in \ell^{(s)}_2((\mathbb{Z}^d)^2).$$

10.1. **Asosiy talablar.** Faraz qilaylik quyidagi shartlar o'rinli bo'lsin

Talab 10.1. (i) $\hat{\varepsilon}(s)$

$$\hat{\varepsilon}(s) = \begin{cases} 3, & s = 0 \\ -\frac{1}{2}, & |\ s\ | = 1 \\ 0, & \textit{qolgan hollarda}, \end{cases} \tag{10.2}$$

ko'rinishga ega bo'lsin.

(ii) $\hat{v}(s)$ *funktsiya* $\mathbb{Z}^d$ *da haqiqiy qiymatli, juft, nomanfiy bo'lib*

$$\lim_{|s|\to\infty} |s|^{3+\rho}\hat{v}(s) = 0, \quad \rho > \frac{1}{2}$$

shartni qanoatlantirsin.

Eslatma 10.2. *10.1 - talab bajarilganda* (10.1) *Hamiltonian* $\ell_2^{(s)}((\mathbb{Z}^d)^2)$ *fazoda o'z-o'ziga qo'shma chegaralangan operatorni aniqlaydi.*

10.2. **Impuls tasvir.** $(\mathbb{T}^d)^2$ orqali $\mathbb{T}^d = (-\pi, \pi]^d$ torning 2-darajali Dekart ko'paytmasini va $L_2^{(s)}((\mathbb{T}^d)^2) \subset L_2((\mathbb{T}^d)^2)$ orqali simmetrik funktsiyalar qism fazosini belgiaymiz.

F.q. $\mathcal{F}_m : L_2((\mathbb{T}^d)^m) \to \ell_2((\mathbb{Z}^3)^m)$ odatdagi Fourier almashtirishi bo'lsin.

$L_2^{(s)}((\mathbb{T}^d)^2)$ qism fazo $\mathcal{F}_2$ guruhga nisbatan invariant, ya'ni $\mathcal{F}_m L_2^{(s)}((\mathbb{T}^d)^2) \subset \ell_2^{(s)}((\mathbb{Z}^d)^2)$.

Ushbu $\mathcal{F}_2$ akslantirishninng $L_2^{(s)}((\mathbb{T}^d)^2)$ fazodagi qismini $\mathcal{F}_2^s$ orqali belgilaymiz. $\mathcal{F}_2^s$ ning aniqlanishidan

$$\mathcal{F}_2^s : L_2^{(s)}((\mathbb{T}^d)^2) \to \ell_2^{(s)}((\mathbb{Z}^d)^2).$$

Ikkki zarrachali Hamiltonian impuls ko'rinishida $L_2^{(s)}((\mathbb{T}^3)^2)$ Gilbert fazosida quyidagi o'z-o'ziga qo'shma chegaralangan operatorni aniqlaydi

$$h = (\mathcal{F}_2^s)^{-1} \widehat{h} \mathcal{F}_2^s,$$

va

$$h = h^0 - v.$$

Bunda h^0 operator $\varepsilon(k_1) + \varepsilon(k_2)$ funktsiyaga ko'paytirish operatori, ya'ni

$$(h^0 f)(k_1, k_2) = (\varepsilon(k_1) + \varepsilon(k_2)) f(k_1, k_2), \; f \in L_2^{(s)}((\mathbb{T}^d)^2)$$

v esa quyidagicha aniqlangan integral operatordir:

$$(vf)(k_\beta, k_\gamma) = (2\pi)^{-\frac{d}{2}} \int\limits_{(\mathbb{T}^d)^2} v(k_\beta - k'_\beta) \delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) f(k'_\beta, k'_\gamma) dk'_\beta dk'_\gamma,$$

$$f \in L_2^{(s)}((\mathbb{T}^d)^2),$$

bunda $\delta(\cdot)$ delta Dirak funksiasini aglatadi,

$\varepsilon(k)$ va $v(k)$, funktsiyalar esa Fourier qatorlari orqali quyidagicha aniqlanadi:

$$\varepsilon(k) = \sum_{j=1}^{d} (1 - cos k^{(j)}), \quad v(k) = (2\pi)^{-d/2} \sum_{s \in \mathbb{Z}^d} \widehat{v}\,(s)\, e^{\mathrm{i}(k,s)},$$

$$(k, s) = \sum\nolimits_{j=1}^{3} k^{(j)} s^{(j)}, \quad k = (k^{(1)}, k^{(2)}, ..., k^{(d)}) \in \mathbb{T}^d, \quad s = (s^{(1)}, s^{(2)}, ..., s^{(d)}) \in \mathbb{Z}^d.$$

10.3. **Hamiltonianlarni fon-Neyman to'g'ri integraliga yoyish. Kvaziimpuls va koordinatalar sistemasi.** Berilgan $m \in \mathbb{N}$ uchun $\widehat{U}_t^m$, $t \in \mathbb{Z}^d$ orqali $\ell_2((\mathbb{Z}^d)^m)$ Hilbert fazoda

$$(\widehat{U}_t^m f)(n_1, n_2, ..., n_m) = f(n_1 + t, n_2 + t, ..., n_m + t), \quad f \in \ell_2((\mathbb{Z}^d)^m)$$

formula bilan aniqlangan unitar operatorlarni belgilaymiz.

$$\widehat{U}_{t+\tau}^m = \widehat{U}_t^m \widehat{U}_\tau^m, \quad t, \tau \in \mathbb{Z}^d,$$

tenglik bajarilishini osongina tekshirish mumkin, va demak $\widehat{U}_t^m, t \in \mathbb{Z}^d$ - $\mathbb{Z}^d$ abel guruhining $\ell_2((\mathbb{Z}^d)^m)$ Hilbert fazodagi unitar tasvirini hosil qiladi.

$\ell_2^{(s)}((\mathbb{T}^d)^2)$ fazo $\widehat{U}_t^m, t \in \mathbb{Z}^d$ guruhga nisbatan invariant, ya'ni

$$\widehat{U}_t^m \ell_2^{(s)}((\mathbb{T}^d)^2) \subset \ell_2^{(s)}((\mathbb{Z}^d)^2).$$

$\widehat{U}_{st}^m$ orqali $\widehat{U}_t^m$ ning $\ell_2^{(s)}((\mathbb{T}^d)^2)$ fazodagi qismini belgilaymiz.

$\mathbb{Z}^3$ abel guruhining $\ell_2((\mathbb{Z}^d)^m)$ Hilbert fazodagi unitar tasviri $\mathcal{F}_m^s-$ Fourier almashtirishlari yordamida $\mathbb{Z}^3$ abel guruhining $L_2^{(s)}((\mathbb{T}^d)^m)$ Hilbert fazodagi unitar tasviri bo'lgan $U_{st}^m = (\mathcal{F}_m^s)^{-1} \widehat{U}_{st}^m \mathcal{F}_m^s$, $t \in \mathbb{Z}^d$ unitar (ko'paytirish) operatorlarga o'tadi:

$$(U_{st}^m f)(k_1, k_2, ..., k_m) = \exp\big(-i(t, k_1 + k_2 + ... + k_m)\big) f(k_1, k_2, ..., k_m), \tag{10.3}$$

$$f \in L_2^{(s)}((\mathbb{T}^d)^m).$$

Berilgan $K \in \mathbb{T}^d$ uchun $\mathbb{F}_K^m$ orqali

$$\mathbb{F}_k^m = \{(k_1, ..., k_{m-1}, K - k_1 - ... - k_{m-1}) \in (\mathbb{T}^3)^m :$$
$$k_1, k_2, ..., k_{m-1} \in \mathbb{T}^d, k - k_1 - ... k_{m-1} \in \mathbb{T}^d\}$$

ni belgilaymiz.

$L_2^{(s)}((\mathbb{T}^d)^m)$ Hilbert fazosini

$$L_2^{(s)}((\mathbb{T}^d)^m) = \int_{K\in\mathbb{T}^d} \oplus L_2^{(s)}(\mathbb{F}_k^m)dk$$

to'gri integralga yoyib, U_{st}^m, $t \in \mathbb{Z}^d$ unitar yoyilmaga mos

$$U_{st}^m = \int_{k\in\mathbb{T}^d} \oplus U_t(k)dk,$$

to'g'ri integral yoyilmani hosil qilamiz, bunda

$$U_t(k) = \exp(-i(t,k))I \quad \text{on} \quad L_2^{(s)}(\mathbb{F}_k^m)$$

va $I = I_{L_2^{(s)}(\mathbb{F}_k^m)} - L_2^{(s)}(\mathbb{F}_k^m)$ Hilbert fazosidagi birlik operator.

Ravshanki, $\widehat{h}$ Hamiltonian (koordinata tasvirida) $\widehat{U}_{st}^2$, $t \in \mathbb{Z}^d$ guruh bilan o'rin almashinuvchi, ya'ni

$$\widehat{U}_{st}^2\widehat{h} = \widehat{h}\widehat{U}_{st}^2, \quad t \in \mathbb{Z}^d$$

Xuddi shunday h (impuls tasvirda) Hamiltonian ham (10.3) formula bilan aniqlangan U_s^2, $s \in \mathbb{Z}^d$ guruh bilan o'rin almashinuvchidir.

Demak, h operatorni

$$L_2^{(s)}(\mathbb{F}_k^2) = \int\limits_{k\in\mathbb{T}^d} \oplus L_2^{(s)}(\mathbb{F}_k^2)dk,$$

yoyilmaga mos

(10.4)
$$h = \int\limits_{k\in\mathbb{T}^d} \oplus\tilde{h}(k)dk$$

to'g'ri integralga yoyish mumkin.

Bunda k, $k \in \mathbb{T}^d$ parametrga *ikki zarrachali kvaziimpuls* va unga mos $\tilde{h}(k),\ k \in \mathbb{T}^d$ operatorlarga *qobiq operatorlar* deyiladi.

Ushbu

$$\pi^{(2)} : (\mathbb{T}^d)^2 \to \mathbb{T}^d, \quad \pi^{(2)}((k_\beta, k_\gamma)) = k_\beta$$

akslantirishni qaraymiz.

$\pi_k^{(2)}$, $k \in \mathbb{T}^d$ orqali $\pi^{(2)}$ akslantirishning $\mathbb{F}_k^2 \subset (\mathbb{T}^d)^2$, dagi qismini belgilaymiz, ya'ni

(10.5)
$$\pi_k^{(2)} = \pi^{(2)}|_{\mathbb{F}_k^2}.$$

$\mathbb{F}_k^2$, $k \in \mathbb{T}^d$ ko'pxillik $\mathbb{T}^d$ ga gomeomorf ekanligini eslatib o'tamiz.

Lemma 10.3. *Teskarisi*

$$(\pi_k^{(2)})^{-1}(k_\beta) = (k_\beta, k - k_\beta)$$

ko'rinishida aniqlangan $\pi_k^{(2)}$, $k \in \mathbb{T}^d : \mathbb{F}_k^2 \subset (\mathbb{T}^d)^2 \to \mathbb{T}^d$ *biyektivdir.*

10.4. **Qobiq operatorlar.** Yuqoridagi (10.4) integral yoyilmadagi $\tilde{h}(k)$, $k \in \mathbb{T}^d$, qobiq operatorlar quyidagi ko'rinishda aniqlangan $h(k)$, $k \in \mathbb{T}^d$, operatorlarga unitar ekvivalent:

(10.6)
$$h(k) = h^0(k) - v.$$

$L_2^e(\mathbb{T}^d) \subset L_2(\mathbb{T}^d)$ juft funktsiyalar qism fazosi bo'lsin. $h^0(k)$ va v bu fazoda quyidagicha ta'sir qiladi:

$$(h^0(k)f)(k_\beta) = \mathcal{E}_k(k_\beta)f(k_\beta), \quad f \in L_2^e(\mathbb{T}^d),$$

bunda

(10.7)
$$\mathcal{E}_k(k_\beta) = \varepsilon(\frac{k}{2} - k_\beta) + \varepsilon(\frac{k}{2} + k_\beta)$$

va

$$(vf)(k_\beta) = (2\pi)^{-\frac{3}{2}} \int\limits_{\mathbb{T}^d} v(k_\beta - k'_\beta)f(k'_\beta)dk'_\beta, \quad f \in L_2^e(\mathbb{T}^d).$$

11. NUQTADA (KONTAKT) TA'SIRLASHUVCHI IKKITA BIR XIL ZARRACHALAR SISTEMASIGA MOS SCHRÖDINGER OPERATORI $h_\mu(k)$ NING SPEKTRAL XOSSALARI (D=1,2)

Faraz qilaylik panjara o'lchami $d = 1, 2$ bo'lsin va zarrachalar ta'sir potensiali quyidagi shartni qanoatlantirsin.

Talab 11.1.

(11.1) $$\hat{v}(s) = \begin{cases} \frac{1}{2}, & \textit{agar} \quad s = 0 \\ 0, & \textit{agar} \ s \neq 0. \end{cases}$$

U holda ikki zarrachali $h(k) \equiv h_\mu(k),\ k \in \mathbb{T}^d$ diskret Shroedinger operatori

$$L_2^{(e)}(\mathbb{T}^d) = \{f \in L_2(\mathbb{T}^d) : f(q) = f(-q)\}$$

fazoda quyidagicha ta'sir qiladi

(11.2) $$h_\mu(k) = h^0(k) - \mu v,$$

bu erda

$$(vf)(k_\beta) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} f(k'_\beta) dk'_\beta, \quad f \in L_2^{(e)}(\mathbb{T}^d),$$

$\mu > 0-$ zarrachalar ta'sirlashuv energiyasi.

Ushbu bo'limning asosiy natijasi quyidagidan iborat:

Teorema 11.2. *Ixtiyoriy $\mu > 0$ va $k \in \mathbb{T}^d$ uchun $h_\mu(k)$ operatorning muhim spektrning tubidan chapda yagona $z_\mu(k)$ xos qiymati mavjud.*

Bundan tashqari $z_\mu(k)$- $\mathbb{T}^d$ da juft va barcha $k \in \mathbb{T}^d \setminus \{0\}$ lar uchun $z_\mu(k) > z_\mu(0)$ tengsizlik o'rinli.

C- kompleks sonlar to'plami bo'lsin.

Ixtiyoriy $k \in \mathbb{T}^d$ va $z \in \mathbf{C} \setminus \sigma_{\text{ess}}(h_\mu(k))$ da aniqlangan

(11.3) $$\Delta_\mu(k, z) = 1 - \mu(2\pi)^{-d} \int\limits_{\mathbb{T}^d} (\mathcal{E}_k(q) - z)^{-1} dq$$

funktsiya($h_\mu(k)$ operatorga mos Fredgolm determinant) ni qaraymiz.

Bu $\Delta_\mu(k, z)$ funktsiya $\mathbb{T}^d \times (\mathbf{C} \setminus \sigma_{\text{ess}}(h_\mu(k)))$ to'plamda haqiqiy qimatli analitik funktsiya bo'lishini eslatib o'tamiz.

Quyidagi lemma Birman-Schwinger prinsipi va Fredholm teoremasidan bevosita kelib chiqadi:

Lemma 11.3. *$z \in \mathbf{C} \setminus \sigma_{ess}(h_\mu(k))$ soni $h_\mu(k)$ operatorning xos qiymati bo'lishi uchun*

$$\Delta_\mu(k, z) = 0$$

bo'lishi zarur va yetarli.

□

Teorema 11.2 ni isbotlash uchun zarur bo'lgan quyidagi lemmani keltiramiz:

Lemma 11.4. *Ixtiyoriy $k \in (-\pi, \pi)^d$ va $z < \mathcal{E}_{\min}(k)$ uchun quyidagi yoyilma o'rinli:*

(i) $d = 1$ bo'lsin. U holda

(11.4) $$\Delta_\mu(k, z) = -\frac{\mu}{2\sqrt{\cos\frac{k}{2}}} (\mathcal{E}_{\min}(k) - z)^{-\frac{1}{2}} + \Delta_\mu^{(20)}(\mathcal{E}_{\min}(k) - z),$$

bunda $\Delta_\mu^{(20)}(\mathcal{E}_{\min}(k) - z) = o(1) \quad$ agar $\quad z \to \mathcal{E}_{\min}(k)$.

(ii) $d = 2$ bo'lsin. U holda

(11.5) $$\Delta_\mu(k, z) = \frac{\mu}{4\pi\sqrt{\cos\frac{k^{(1)}}{2} \cos\frac{k^{(2)}}{2}}} \ln(\mathcal{E}_{\min}(k) - z) + \Delta_\mu^{(20)}(\mathcal{E}_{\min}(k) - z),$$

bunda $\Delta_\mu^{(20)}(\mathcal{E}_{\min}(k) - z) = o(1) \quad$ agar $\quad z \to \mathcal{E}_{\min}(k)$.

Proof. $\Delta_\mu(k, z)$ funktsiyani quyidagicha tasvirlymiz

(11.6) $$\Delta_\mu(k, z) = 1 - \frac{\mu}{(2\pi)^d} D(k, z),$$

bunda

(11.7) $$D(k, z) = \int\limits_{\mathbb{T}^d} \frac{dq}{\mathcal{E}_k(q) - z}.$$

Har bir $k \in (-\pi, \pi)^d$ da $\mathcal{E}_k(q)$ funktsiya $q_k = \frac{k}{2}$ nuqtada aynimagan minimumga ega. Shu sababli parametrga bogliq Mors lemmasiga ko'ra ([32] ga q.) koordinatalar boshining $\gamma > 0$ radiusli $W_\gamma(0)$ atrofini q_k nuqtaning $\tilde{W}(q_k)$ atrofiga o'tkazuvchi o'zaro bir qiymatli regulyar $q = \varphi(k; t)$ akslantirish mavjud bo'lib,

(11.8) $$\mathcal{E}_k(\varphi(k; t)) = t^2 + \mathcal{E}_{\min}(k)$$

$\varphi(k; 0) = 0$ tenliklar o'rinli bo'ladi.

Bu $q = \varphi(k; t)$ akslantirishning Yakobiani $J(\varphi(k; t))$ panjara o'lchami $d = 1$ bo'lsa

$$J(\varphi(k, 0)) = \frac{1}{\sqrt{\cos \frac{k}{2}}}$$

va $d = 2$ bo'lsa

$$J(\varphi(k, 0)) = \frac{1}{\sqrt{\cos \frac{k^{(1)}}{2} \cos \frac{k^{(2)}}{2}}}, \; k = (k^{(1)}, k^{(2)}) \in \mathbb{T}^2$$

tengliklarni qanoatlantiradi.

$D(k, z)$ ni quyidagi funktsiyalar yig'indisi ko'rinishida tasvirlaymiz

(11.9) $$D(k, z) = D_1(k, z) + D_2(k, z),$$

bunda

(11.10) $$D_1(k, z) = \int_{\tilde{W}(q_k)} \frac{dq}{\mathcal{E}_k(q) - z},$$

va

$$D_2(k, z) = \int_{\mathbb{T}^d \setminus \tilde{W}(q_k)} \frac{dq}{\mathcal{E}_k(q) - z}.$$

Har bir $k \in (-\pi, \pi)^d$ uchun $\mathcal{E}_k(\cdot)$ funktsiya $\mathbb{T}^d \setminus \tilde{W}(q_k)$ kompakt to'plamda uzluksiz va $q_k \in U_\delta(0)$ nuqtada aynimagan minimumga ega bo'lganligi uchun shunday $M = const > 0$ son topilib

$$\inf_{q \in \mathbb{T}^d \setminus \tilde{W}(q_k)} \mathcal{E}_k(q) \geq M$$

tengsizlik bajariladi.

(11.10) tenglikning o'ng qismidagi integralda $q = \varphi(k; t)$ almashtirish olamiz va (15.12) tenglikdan foydalanib

(11.11) $$D_1(k, z) = \int_{W_\gamma(0)} \frac{J(\varphi(k; t))dt}{t^2 + \mathcal{E}_{\min}(k) - z}$$

ga ega bo'lamiz.

$q = \varphi(k; t)$ akslantirishning regulyarligidan ([32] ga q.)

(11.12) $$J(\varphi(k; t)) = \sum_{j=0}^{\infty} C_j(k) t^{2j}$$

tenglikni hosil qilamiz, bunda $C_j(k)-$ $(-\pi, \pi)^d$ dagi analitik funktsiya.

$d = 1$ bo'lsin.

(11.11) ning o'ng qismidagi integral ostidagi funktsiya juftligidan

(11.13) $$D_1(k, z) = 2 \int_0^\gamma \frac{J(\varphi(k; t))dt}{t^2 + \mathcal{E}_{\min}(k) - z}$$

tenglikni hosil qilamiz.

(11.12) tenglikka ko'ra ixtiyoriy $k \in (-\pi, \pi)^d$ uchun shunday $C > 0$ son topiladiki,

(11.14) $$| \, J(\varphi(k; t)) - J(\varphi(k; 0)) \, | \leq C \mid t \mid^2$$

tengsizlik bajariladi.

$D_1(k, z)$ funktsiyani

(11.15) $$D_1(k, z) = 2 \int_0^\gamma \frac{J(\varphi(k; 0))dt}{t^2 + \mathcal{E}_{\min}(k) - z} + 2 \int_0^\gamma \frac{J(\varphi(k; t)) - J(\varphi(k; 0))dt}{t^2 + \mathcal{E}_{\min}(k) - z}$$

ko'rinishda tasvirlab olamiz.

Tekshirish mumkinki,

$$\int_0^\gamma \frac{dr}{r^2+\mathcal{E}_{\min}(k)-z} \to \frac{\pi}{2} \ agar \ \mathcal{E}_{\min}(k)-z \to 0. \tag{11.16}$$

Shunday qilib (11.16), (11.15),(11.14), (11.9) va (11.6) munosabatlardan (11.4) ga kelamiz.

$d=2$ bo'lsin. (15.13)tenglikning o'ng qismidagi integralda $t=r\omega$ qutb koordinatalar sistemasiga o'tib $D_1(k,z)$ funktsiyani

$$D_1(k,z)=\int_0^\gamma \frac{rF(k,r)dr}{r^2+\mathcal{E}_{\min}(k)-z} \tag{11.17}$$

ko'rinishga keltiramiz,

bunda$F(k,r)=\int_{\mathbb{S}^1} J(\varphi(k;r\omega))d\omega$, va $\mathbb{S}^1-\mathbb{R}^2$ dagi birlik aylana , $d\omega$ esa shu fazodagi birlik aylana elementi.

$F(k,r)$ funktsiyaning aniqlanishidan ixtiyoriy $k\in(-\pi,\pi)^d$ uchun shunday $C>0$ son topiladiki,

$$\mid F(k,r)-F(k,0)\mid \le C\mid r\mid^2 \tag{11.18}$$

munosabat bajariladi.

$D_1(k,z)$ funktsiyani

$$D_1(k,z)=F(k;0)\int_0^\gamma \frac{rdr}{r^2+\mathcal{E}_{\min}(k)-z}+ \tag{11.19}$$

$$+\int_0^\gamma \frac{r(F(k;r)-F(k;0))}{r^2+\mathcal{E}_{\min}(k)-z}dr$$

ko'rinishga keltiramiz.

(11.18) tengsizlikka ko'ra

$$\int_0^\gamma \frac{r(F(k;r)-F(k;0))}{r^2+\mathcal{E}_{\min}(k)-z}dr \le C\int_0^\gamma \frac{r^3dr}{r^2+\mathcal{E}_{\min}(k)-z}. \tag{11.20}$$

Qaralayotgan

$$\int_0^\gamma \frac{rdr}{r^2+\mathcal{E}_{\min}(k)-z} \quad \text{va} \quad \int_0^\gamma \frac{r^3dr}{r^2+\mathcal{E}_{\min}(k)-z}$$

integrallar uchun

$$\int_0^\gamma \frac{rdr}{r^2+\mathcal{E}_{\min}(k)-z} \to \frac{1}{2}\ln(\mathcal{E}_{\min}(k)-z) \ agar \ \mathcal{E}_{\min}(k)-z\to 0 \tag{11.21}$$

$$\int_0^\gamma \frac{r^3dr}{r^2+\mathcal{E}_{\min}(k)-z} \to 0 \, agar \ \mathcal{E}_{\min}(k)-z\to 0 \tag{11.22}$$

munosabatlar o'rinli.

Shunday qilib (11.18), (15.14), (11.15),(11.7) va (11.6) lardan (11.5) tenglikka ega bo'lamiz.

□

Lemma 11.5. *Ixtiyoriy $z<\mathcal{E}_{\min}(k)$ uchun $\Delta_\mu(k,z)=\Delta_\mu(k^{(1)},k^{(2)},...,k^{(d)};z)$ funktsiya har bir $k^{(i)}\in(-\pi,\pi], i=\overline{1,d}$, bo'yicha juft va $k^{(i)}\in(0,\pi], i=\overline{1,d}$, bo'yicha monoton o'suvchi.*

11.2 teoremaning isboti.

Ixtiyoriy $k\in\mathbb{T}^d$ va $\mu>0$ uchun $\Delta_\mu(k,\cdot)$ funktsiya $\mathbf{C}\backslash\sigma_{\text{ess}}(h_\mu(k))$ to'plamda analitik va $z\in(-\infty,\mathcal{E}_{\min}(k))$ bo'yicha hosilasi manfiydir, ya'ni

$$\frac{\partial\Delta_\mu(k,z)}{\partial z}=-\mu(2\pi)^{-d}\int\limits_{\mathbb{T}^d}(\mathcal{E}_k(q)-z)^{-2}dq<0.$$

Bundan $\Delta_\mu(k,.)$ funktsiyaning $(-\infty,\mathcal{E}_{\min}(k))$ oraliqda monoton kamayuvchi ekanligiga kelamiz.

$\Delta_\mu(k,z)$ ning aniqlanishidan va 11.4 lemmaga ko'ra

$$\lim_{z\to-\infty}\Delta_\mu(k,z)=1$$

va

$$\lim_{z\to\mathcal{E}_{\min}(k))-}\Delta_\mu(k,z)=-\infty$$

tengliklar o'rinli.

Bu mulohazalardan $\Delta_\mu(k,.)$ funktsiyaning $(-\infty,\mathcal{E}_{\min}(k))$ da yagona noli borligiga kelamiz.

Lemma 11.3 ga ko'ra ixtiyoriy $k \in \mathbb{T}^d$ va $\mu > 0$ uchun $h_\mu(k)$ operatorning muhim spektrning tubidan chapda yagona $z_\mu(k)$ xos qiymati mavjudligiga kelamiz.

Lemma 11.5 dan, ixtiyoriy $0 < \mu_1 < \mu_2$ uchun $\Delta_{\mu_1}(k,z) > \Delta_{\mu_2}(k,z)$ ekanligidan va $\Delta_\mu(-k,z) = \Delta_\mu(k,z)$ tenglikdan $z_\mu(k) > z_\mu(0)$, $z_{\mu_1}(k) > z_{\mu_2}(0)$ va $z_\mu(-k) = z_\mu(k)$ munosabatlarni hosil qilish mumkin.□

Lemma 11.6. *Ixtiyoriy $k \in U_\delta(0)$ uchun shunday $\delta(k) > 0$ son topiladiki, barcha $z \in V_{\delta(k)}(z_\mu(k))$, bunda $V_{\delta(k)}(z_\mu(k))$ - $z_\mu(k)$ ning $\delta(k)$- atrofi, quyidagi yoyilma o'rinli bo'ladi*

$$\Delta_\mu(k,z) = C_1(k)(z - z_\mu(k))\hat{\Delta}_\mu(k,z).$$

Bu erda $C_1(k) \neq 0$ va $\hat{\Delta}_\mu(k,z)$ $V_{\delta(k)}(z_\mu(k))$ da uzluksiz va $\hat{\Delta}_\mu(k, z_\mu(k)) \neq 0$.

Proof. $z_\mu(k) < \mathcal{E}_{\min}(k)$, $k \neq 0$ bo'lganligi uchun $\Delta_\mu(k,z)$ funktsiyani quyidagicha tasvirlash mumkin:

$$\Delta_\mu(k,z) = \sum_{n=1}^{\infty} C_n(k)(z - z_\mu(k))^n, \quad z \in V_{\delta(k)}(z_\mu(k)),$$

bunda

$$C_1(k) = \frac{\mu m^{3/2}}{\sqrt{2}\pi} \frac{1}{2\sqrt{E_{\min}(k) - z_\mu(k)}} \neq 0, \quad k \neq 0.$$

Bundan tashqari $\hat{\Delta}_\mu(k,z)$ funktsiya $V_{\delta(k)}(z_\mu(k))$ atrofda uzluksiz. $z_\mu(k)$, $k \neq 0$ son $\Delta_\mu(k,z) = 0$, $z \leq E_{\min}^{(}k)$, tenglamaning oddiy yechimi ekanligidan $\hat{\Delta}_\mu(k, z_\mu(k)) \neq 0$.

□

12. Yana virtual sath haqida (d=3)

Panjara o'lchami $d = 3$ bo'lsin va zarrachalar ta'sir potensiali $\hat{v}(s)-$ 10.1 talabni qanoatlantirsin.

Bu paragrafda ushbu talablar asosida panjaradagi ikki zarrachali operator $h(0)$ uchun virtual sath (*threshold resonance*) tushunchasini kiritamiz.

3.1 talabga ko'ra $v(p)$ uzluksiz funktsiyaning $\{\hat{v}(s)\}_{s\in\mathbb{Z}^3}$ Fourier koeffitsenlari $\ell_2(\mathbb{Z}^3)$ ning elementi bo'ladi va

$$v(p) = (2\pi)^{-\frac{3}{2}} \sum_{s\in\mathbb{Z}^3} \hat{v}(s)e^{\mathrm{i}(p,s)}, \tag{12.1}$$

tenglikni $(2\pi)^{-\frac{3}{2}} \sum_{s\in\mathbb{Z}^3} \hat{v}(s)e^{\mathrm{i}(p,s)}$ funktsiyaning $L_2(\mathbb{T}^3)$ fazodagi $v(p)-$ uzluksiz vakilga ega bo'lgan tasviri deb tushunamiz.

(10.7) bilan aniqlangan $\mathcal{E}_k(q)$ funktsiya $q = \frac{k}{2}$ nuqtada yagona aynimagan minimumga ega bo'lganligi uchun ixtiyoriy $k \in (-\pi,\pi)^3$ va $z \leq \mathcal{E}_{\text{min}}(k)$ uchun

$$G(p,q;k,z) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{v^{\frac{1}{2}}(p-t)v^{\frac{1}{2}}(t-q)dt}{\mathcal{E}_k(t) - z} \tag{12.2}$$

integral cheklidir, bu erda

$$v^{\frac{1}{2}}(p) = (2\pi)^{-\frac{3}{2}} \sum_{s\in\mathbb{Z}^3} \hat{v}^{\frac{1}{2}}(s)e^{\mathrm{i}(p,s)} \tag{12.3}$$

orqali $v^{\frac{1}{2}}$ operatorning yadrosi belgilangan.

Shuning uchun $L_2^e(\mathbb{T}^3)$ fazoda $G(p,q;k,z)$ yadroli $G(k,z)$ integral operatorni aniqlash mumkin.

Eslatma 12.1. *Ravshanki, (taqqoslang* [61]*), h operator $z \leq \mathcal{E}_0(q) = 0$ xos qiymatga ega, ya'ni, $\mathrm{Ker}(h(0) - \lambda I) \neq 0$ bo'lishi uchun $G(0,z)$ kompakt operator $C(\mathbb{T}^d)$ da 1 xos qiymatga ega va shunday $\psi \in \mathrm{Ker}(I - G(0,z))$ funktsiya topilib, deyarli hamma $p \in \mathbb{T}^d$ larda*

$$f(p) = \frac{(v^{\frac{1}{2}}\psi)(p)}{\mathcal{E}_0(q) - z} \quad \textit{a.e.} \quad p \in \mathbb{T}^3,$$

ko'rinishda berilgan $f, f \in \mathrm{Ker}(h(0) - \lambda I)$ funktsiya $L^2(\mathbb{T}^d)$ ga qarashli bo'lishi zarur va etarli.

Eslatma 12.2. *Agar* $z < \mathcal{E}_0(0) = 0$ *bo'lsa,*

$$\dim \mathrm{Ker}(h(0) - zI) = \dim \mathrm{Ker}(I - G(0,z))$$

va

$$\mathrm{Ker}(h(0) - zI) = \{f \mid f(\cdot) = \frac{(v^{\frac{1}{2}}\psi)(\cdot)}{\varepsilon(\cdot) - z}\,,\, \psi \in \mathrm{Ker}(I - G(0,z))\}$$

tengliklar bajariladi.

Agar $z = 0$ *bo'lsa,* (12.2) *tenglik buzilishi mumkin.*

Shu sababli (12.2) *tenglik quyidagi*

$$\dim \mathrm{Ker}(h(0)) \leq \dim \mathrm{Ker}(I - G(0,0))$$

tenglikka almashadi.

Ta'rif 12.3. *Agar 1 soni* $G(0,0)$ *operator uchun oddiy xos qiymat bo'lib, mos normalangan* ψ *xos funsiya*

$$\frac{(v^{\frac{1}{2}}\psi)(\cdot)}{\varepsilon(\cdot) - \varepsilon(0)} \notin L^2(\mathbb{T}^3)$$

ni qanoatlantirsa, ya'ni

$$1 \leq \dim \mathrm{Ker}(G(z) + I) \geq \dim \mathrm{Ker}(h - zI) + 1$$

bo'lsa, $h(0)$ *operator 0 da virtual sathga ega (0 energiyali rezonansga ega) deyiladi.*

Eslatma 12.4. $h(0)$ *operator 0 energiyali rezonansga ega bo'lishi uchun* 1 *soni* $G(0,0)$ *operator uchun oddiy xos qiymat va unga mos* $\psi \in L_2^e(\mathbb{T}^3)$ *xos funktsiya* $\int v^{1/2}(p)\psi(p)dp \neq 0$ *shartni qanoatlantirishi zarur va yetarli.*

Bizning asosiy natijalarimizdan biri quyidagi teoremadan iborat:

Teorema 12.5. *F.q. 10.1 talab bajarilsin va* $h(0)$ *operator 0 energiyali rezonansga ega bo'lsin. U holda ixtiyoriy* $k \in U_\delta^0(0)$ *uchun* $h(k)$ *operator muhim spektr tubidan pastda yagona musbat* $z(k)$ *xos qiymatga ega. Bundan tashqari agar* $z(0) = 0$ *deb olsak,* $z(k)$ *funktsiya* $U_\delta(0)$ *da analitik bo'ladi.*

Quyidagi lemma $\mathbb{Z}^3$ panjaradagi ikki zarrachali Schrödinger operatori uchin Birman-Schwinger prinsipini ifodalaydi:

Lemma 12.6. $G(k,z),\ k \in U_\delta(0),\ z \leq \mathcal{E}_{\min}(k)$ *operator* $L_2(\mathbb{T}^3)$ *fazoda musbat,* Σ_1 *sinfga qarashli,* z *bo'yicha* $z = \mathcal{E}_{\min}(k)$ *gacha chapdan uzluksiz va quyidagi tenglik o'rinlidir*

$$n(-\mathcal{E}_{\min}(k), -h(k)) = n(1, G(k, \mathcal{E}_{\min}(k))).$$

□

12.5 Teoremaning isboti. Teorema shartlariga ko'ra

(12.4) $$G_0\psi = v^{\frac{1}{2}} r_0(0,0) v^{\frac{1}{2}}\psi = \psi$$

tenglama no'lmas $\psi \in L_2^e(\mathbb{T}^3)$ yechimga ega.

U holda

$$(\psi,\psi) = (v^{\frac{1}{2}} r_0(0,0) v^{\frac{1}{2}}\psi, \psi) = (r_0(0,0)\varphi, \varphi) = \int\limits_{\mathbf{T}^3} \frac{|\varphi(p)|^2}{\mathcal{E}_0(p)} dp,$$

bunda $\varphi = v^{1/2}\psi \in C(\mathbb{T}^3)$.

O'z navbatida

$$(v^{\frac{1}{2}} r_0(k, \mathcal{E}_{\min}(k)) v^{\frac{1}{2}}\psi, \psi) = \int\limits_{\mathbf{T}^3} \frac{|\varphi(p)|^2 dp}{\mathcal{E}_k(p) - \mathcal{E}_{\min}(k)}$$

$$= \int\limits_{\mathbf{T}^3} \frac{|\varphi(p)|^2 dp}{\sum_{i=1}^3 2(1 - cos p_i) cos(k_i/2)} > \int\limits_{\mathbf{T}^3} \frac{|\varphi(p)|^2 dp}{\sum_{i=1}^3 2(1 - cos p_i)}$$

$$= \int\limits_{\mathbf{T}^3} \frac{|\varphi(p)|^2}{\mathcal{E}_0(p)} dp = (\psi,\psi)$$

ga kelamiz.

7.5 ning aniqlanishidan $n(1, v^{\frac{1}{2}} r_0(k, \mathcal{E}_{\min}(k)) v^{\frac{1}{2}}) > 1$ va nihoyat Birman-Schwinger prinsipidan $h(k), k \neq 0$ opetatorning $\mathcal{E}_{\min}(k)$ dan chapda xos qiymati mavjudligiga kelamiz.

Endi ixtiyoriy $k \in \mathbb{T}^3$ va nolmas $f \in L_2^e(\mathbb{T}^3)$ lar uchun

$$(h(k)f, f) > 0$$

ni ko'rsatamiz.

Quyidagi

$$(h(k)f,f) = \int\limits_{\mathbb{T}^3} \varepsilon(\frac{k}{2}+q) \mid f(q) \mid^2 dq \tag{12.5}$$

$$+ \int\limits_{\mathbb{T}^3} \varepsilon(\frac{k}{2}-q) \mid f(q) \mid^2 dq - \int\limits_{\mathbb{T}^3}\int\limits_{\mathbb{T}^3} v(p-q) f(q)\overline{f(p)} dq dp \tag{12.6}$$

tenglikning o'ng qismidagi integrallarda o'zgaruvchilarni almashtiramiz va f funktsiyaning juftligidan foydalanib

$$(h(k)f,f) = (h(0)g,g) > 0$$

tengsizlikni hosil qilamiz, bunda $g(q) = f(\frac{k}{2}-q)$.

Va natijada $h(0)$ operatorning nomanfiyligidan barcha $k \in \mathbb{T}^3 \setminus \{0\}$ lar uchun $h(k) > 0$ munosabat o'rinlidir.

$G(k,0),\ k \in U_\delta(0)$ operator qiymatli funktsiya sifatida $U_\delta(0)$ da analitik. Shuning uchun (**??** ga q.) $z(k)$ xos qiymatning ham $k \in U_\delta(0)$ da analitiklgiga kelamiz. □

G_1 orqali quyidagi yadroli operatorni belgilaymiz

$$G_1(p,p') = -\frac{1}{8\pi} v^{\frac{1}{2}}(p) v^{\frac{1}{2}}(p'). \tag{12.7}$$

Quyidagi tasdiq o'rinlidir

Lemma 12.7. *3.1 talab o'rinli bo'lsin. U holda har bir $k \in U_\delta(0)$ uchun quyidagi yoyilma o'rinli*

$$G(k,0) = G(0,0) + |k|G_1 + |k|^2 G_2(k), \tag{12.8}$$

bunda $G_2(k)$ operator $k \in U_\delta(0)$ da uzluksiz.

Proof. Har bir $k \in U_\delta(0)$ da $\mathcal{E}_k(q)$ funktsiya $q_k = \frac{k}{2}$ nuqtada aynimagan minimumga ega.

Parametrga bogliq Mors lemmasiga ko'ra ([32] ga q.) shunday koordinatalar boshining $\gamma > 0$ radiusli $W_\gamma(0)$ atrofini q_k nuqtaning $\tilde{W}(q_k)$ atrofiga o'tkazuvchi o'zaro bir qiymatli regulyar $q = \varphi(k;t)$ akslantirish mavjud bo'lib,

$$\mathcal{E}_k(\varphi(k;t)) = t^2 + \mathcal{E}_{\min}(k) \tag{12.9}$$

$\varphi(k;0) = 0$ tenliklar o'rinli bo'ladi.

Bu $q = \varphi(k;t)$ akslantirishning Yakobiani $J(\varphi(k;t))$ quyidagi tenglikni qanoatlantiradi:

$$J(\varphi(k,0)) = \frac{1}{\sqrt{cos\frac{k^{(1)}}{2} cos\frac{k^{(2)}}{2} cos\frac{k^{(3)}}{2}}}. \tag{12.10}$$

$\mathcal{E}_0(\cdot)$ funktsiya $t = 0$ nuqtada aynimagan minimumga ega ega bo'lganligi uchun integral belgisi ostida limitga o'tish haqidagi Lebeg teoremasiga ko'ra barcha $p, q \in \mathbb{T}^3$ lar uchun quyidagi chekli limit mavjud:

$$G(p,q;0,0) = \lim_{k \to 0} G(p,q;k,0).$$

Barcha $p, q \in \mathbb{T}^3$ lar uchun shunday p va q lardan bogliqsiz C musbat son topilib,

$$|G(p,q;k,0) - G(p,q;0,0)| \le C|k|, \tag{12.11}$$

$$\Big|\frac{\partial}{\partial |k|} G(p,q;k,0) - \frac{\partial}{\partial |k|} G(p,q;0,0)\Big| < C|k|^2, \quad k \in U_\delta(0) \tag{12.12}$$

tengsizliklar bajariladi.

Haqiqatan. $G(p,q;\cdot,0)$ funktsiyani

$$G(p,q;k,0) = G_1(p,q;k,0) + G_2(p,q;k,0)$$

ko'rinishida tasvirlaymiz,

bunda

$$G_1(p,q;k,0) = \frac{1}{(2\pi)^3} \int\limits_{\tilde{W}(q_k)} \frac{v^{\frac{1}{2}}(p-t) v^{\frac{1}{2}}(t-q) dt}{\mathcal{E}_k(t)}, \quad k \in U_\delta(0), \tag{12.13}$$

va

$$G_2(p,q;k,0) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3 \setminus \tilde{W}(q_k)} \frac{v^{\frac{1}{2}}(p-t)v^{\frac{1}{2}}(t-q)dt}{\mathcal{E}_k(t)} \quad k \in U_\delta(0).$$

Har bir $k \in U_\delta(0)$ uchun $\mathcal{E}_k(\cdot)$ funktsiya $\mathbb{T}^3 \setminus \tilde{W}(q_k)$ kompakt to'plamda uzluksiz va $q_k \in U_\delta(0)$ nuqtada aynimagan minimumga ega bo'lganligi uchun shunday $M = const > 0$ son topilib

$$\inf_{q \in \mathbb{T}^3 \setminus \tilde{W}(q_k)} \mathcal{E}_k(q) \geq M$$

tengsizlik bajariladi.

Bundan esa barcha $p, q \in \mathbb{T}^3$ lar uchun $p, q \in \mathbb{T}^3$ lardan bog'liqsiz $C > 0$ con topilib,

$$|G_2(p,q;k,0) - G_2(p,q;0,0)| \leq Ck^2, \ k \in U_\delta(0) \tag{12.14}$$

kelib chiqadi.

Quyidagiga egamiz

$$G_1(p,q;k,0) - G_1(p,q;0,0) \tag{12.15}$$

$$= -\frac{1}{(2\pi)^3} \int\limits_{\tilde{W}(q_k)} \frac{(\mathcal{E}_0(t) - \mathcal{E}_k(t))v^{\frac{1}{2}}(p-t)v^{\frac{1}{2}}(t-q)dt}{\mathcal{E}_k(t)\mathcal{E}_0(t)}. \tag{12.16}$$

(15.11) integralda $q = \varphi(k;t)$ almashtirish olamiz va (15.12) tenglikka ko'ra

$$G_1(p,q;k,0) - G_1(p,q;0,0) \tag{12.17}$$

$$= -\frac{\mathcal{E}_{\min}(k)}{(2\pi)^3} \int\limits_{W_\gamma(0))} \frac{v^{\frac{1}{2}}(p - \varphi(k;t))v^{\frac{1}{2}}(\varphi(k;t) - q)J(\varphi(k;t))}{t^2(t^2 + \mathcal{E}_{\min}(k))} dt \tag{12.18}$$

ga ega bo'lamiz.

(15.13) integralda $t = r\omega$ sferik koordinatalar sistemasiga o'tib, uni quyidagi ko'rinishga keltiramiz

$$G_1(p,q;k,0) - G_1(p,q;0,0) = -\frac{\mathcal{E}_{\min}(k)}{(2\pi)^3} \int_0^\gamma \frac{F(p,q;k,r)}{r^2 + \mathcal{E}_{\min}(k)} dr, \tag{12.19}$$

bunda

$$F(p,q;k,r) = \int_{\Omega_2} v^{\frac{1}{2}}(p - \varphi(k,r\omega))v^{\frac{1}{2}}(\varphi(k,r\omega) - q)J(\varphi(k,r\omega))d\omega,$$

Ω_2- esa $\mathbb{R}^3$ dagi birlik shar va $d\omega$ shu fazodagi birlik shar elementi.

Barcha $p, q \in \mathbb{T}^3$ lar uchun ulardan bog'liqsiz $C > 0$ son topilib

$$|F(p,q;k,r) - F(p,q;0,0)| \leq C(r^\theta + |k|^2) \tag{12.20}$$

tengsizlik bajarilishini ko'rsatamiz.

$v^{\frac{1}{2}}(\cdot)$ funktsiyaning $\mathbb{T}^3$ da juftligidan va (12.3) tenglikka ko'ra

$$|v^{\frac{1}{2}}(p-t)v^{\frac{1}{2}}(t-q) - v^{\frac{1}{2}}(p)v^{\frac{1}{2}}(q)| \tag{12.21}$$

$$\leq \frac{1}{(2\pi)^3} \sum_{s \in \mathbb{Z}^3} |\hat{v}(s)||e^{\mathrm{i}(p+q,s)}||e^{-2\mathrm{i}(t,s)} - 1|. \tag{12.22}$$

Ma'lumki, ixtiyoriy $0 < \theta \leq 1$ son uchun $|e^{-2\mathrm{i}(t,s)} - 1| \leq C|t|^\theta |s|^\theta$, $p \in \mathbb{T}^3$, $s \in \mathbb{Z}^3$ tengsizlik bajariladi.

Bundan 3.1 talab va (12.21) tengsizlikka ko'ra $p, q \in \mathbb{T}^3$ lardan bog'liqsiz shunday $C > 0$ son topilib,

$$|v^{\frac{1}{2}}(p-t)v^{\frac{1}{2}}(t-q) - v^{\frac{1}{2}}(p)v^{\frac{1}{2}}(q)| \leq C|t|^\theta, \quad \frac{1}{2} < \theta \leq 1, \tag{12.23}$$

o'rinli ekanligini hosil qilamiz.

$\varphi(k,\cdot)$ funktsiyaning regulyrligidan va (12.10)ga ko'ra

$$|J(\varphi(k,r)) - J(\varphi(k,0))| \leq C|r|^2 \quad \text{va} \quad |J(\varphi(k,0)) - J(\varphi(0,0))| \leq C|k|^2.$$

Bundan

$$|F(p,q;k,r) - F(p,q;0,0)| \leq |F(p,q;k,r) - F(p,q;k,0)|$$
$$+ |F(p,q;k,0) - F(p,q;0,0)|$$
$$\leq \int_{\Omega_2} |v^{\frac{1}{2}}(p - \varphi(k,r\omega))v^{\frac{1}{2}}(\varphi(k,r\omega) - q) - v^{\frac{1}{2}}(p)v^{\frac{1}{2}}(q)||J(\varphi(k,0))|d\omega$$

$$+\int_{\Omega_2}|v^{\frac12}(p-\varphi(k,r\omega))v^{\frac12}(\varphi(k,r\omega)-q)||J(\varphi(k,r\omega))-J(\varphi(k,0)|d\omega+$$

$$\int_{\Omega_2}|v^{\frac12}(p)||v^{\frac12}(q)||J(\varphi(k,0))-J(\varphi(0,0))|d\omega\le C(r^\theta+|k|^2).$$

Barcha $p,q\in\mathbb{T}^3$ lar uchun $G_1(p,q;k,0)-G_1(p,q;0,0)$ ayirma funktsiyani

(12.24)
$$G_1(p,q;k,0)-G_1(p,q;0,0)=-\frac{4\pi F(p,q;0,0)}{(2\pi)^3}\int_0^\gamma\frac{\mathcal{E}_{\min}(k)dr}{r^2+\mathcal{E}_{\min}(k)}-$$
$$\frac{1}{(2\pi)^3}\int_0^\gamma\frac{\mathcal{E}_{\min}(k)(F(p,q;k,r)-F(p,q;0,0))}{r^2+\mathcal{E}_{\min}(k)}dr$$

ko'rinishda tasvirlash mumkin.

(15.15) tengsizlikdan foydalansak

(12.25)
$$\int_0^\gamma\frac{F(p,q;k,r)-F(p,q;0,0)}{r^2+\mathcal{E}_{\min}(k)}dr\le C\int_0^\gamma\frac{r^\theta+|k|^2}{r^2+\mathcal{E}_{\min}(k)}dr$$

ga ega bo'lamiz.

Quyidagi

(12.26)
$$\mathcal{E}_{\min}(k)=\frac14k^2+O(|k|^4)\quad\text{as}\quad k\to0$$

asimptotikaga ko'ra

(12.27)
$$\frac1k\left(\int_0^\gamma\frac{\mathcal{E}_{\min}(k)dr}{r^2+\mathcal{E}_{\min}(k)}-\frac14\int_0^\gamma\frac{k^2dr}{r^2+\frac14k^2}\right)\to0\ as\ k\to0.$$

Qaralayotgan

$$\int_0^\gamma\frac{|k|}{r^2+\frac14k^2}dr\quad\text{va}\quad\int_0^\gamma\frac{r^\theta+|k|^2}{r^2+\frac14k^2}dr$$

integrallar uchun

(12.28)
$$\int_0^\gamma\frac{|k|}{r^2+\frac14k^2}dr\to\pi\ \ |k|\to0\quad\text{va}\quad\int_0^\gamma\frac{|k|(r^\theta+|k|^2)}{r^2+\frac14k^2}dr\to0\ \ |k|\to0$$

munosabatlar o'rinli.

(12.28), (15.16) tenglik va (12.25) tengsizliklardan foydalanib

(12.29)
$$\frac{\partial}{\partial|k|}G_1(p,q;0,0)=\lim_{|k|\to0+}\frac{G_1(p,q;k,0)-G_1(p,q;0,0)}{|k|}=-\frac{1}{8\pi}F(p,q;0,0).$$

tenglikni hosil qilamiz.

Bundan shunday $C>0$ son topilib, ($p,q\in\mathbb{T}^3$ lardan bog'liqsiz)

(12.30)
$$\big|G_1(p,q;k,0)-G_1(p,q;0,0)\big|<C|k|,\quad k\in U_\delta(0)$$

tengsizlik bajarilishi kelib chiqadi.

O'z navbatida (12.14) va (12.30) dan $G_1(p,q;\cdot,0)$ funktsiya $|k|=0$ nuqtada o'ngdan uzluksiz hosilaga ega va barcha $p,q\in\mathbb{T}^3$ lar uchun

$$\frac{\partial}{\partial|k|}G(p,q;0,0)=-\frac{1}{8\pi}v^{\frac12}(p)v^{\frac12}(q)$$

tehglik o'rinlidir.

(12.14) va (12.30) tengsizliklardan (12.11) ni hosil qilamiz.

Yuqoridagiga o'xshash mulohazalar bilan (15.9) tengsizlikni isbotlash mumkin.

Shunday qilib (12.8) ga kelamiz. □

Lemma 12.8. *10.1 talab bajarilsin. U holda har bir $z<0$ uchun quyidagi yoyilma o'rinli*

$$v^{\frac12}r_0(0,-z)v^{\frac12}=G(0,0)+G_1(-z)^{\frac12}+(-z)^{\frac12+\theta_1}G_2(z),\ \ \theta_1<\theta,$$

bunda $G_2(z)-$ operator $z<0$ bo'yicha uzluksiz.

The Lemma 12.8 can be proven by the same way as Lemma 12.7. □

Lemma 12.9. *3.1 talab bajarilsin.*

$a)$ *F.q.* 0 *nuqta* $h(0)$ *operator uchun regulyar nuqta bo'lsin. U holda har bir* $k \in U_\delta(0)$ *uchun* $w(k,0)$ *chegaralangan operator bo'ladi.*

$b)$ *F.q.* $h(0)$ *operator nol energiyali resonansga ega bo'lsin.*

U holda quyidagi tenglik o'rinlidir:

(i)

$$w(k,0) = \frac{8\pi(\cdot,\psi)\psi}{\varphi^2(0)||\psi||^2|k|} + \hat{w}(k,0),\ k \in U_\delta^0(0) \tag{12.31}$$

bunda $\varphi(0) = (v^{\frac{1}{2}}, \psi),$ *va* $\hat{w}(k,0)$ *operator* $k \in U_\delta(0)$ *bo'yicha uzluksiz;*

(ii)

$$w^{\frac{1}{2}}(k,0) = \frac{2\sqrt{2\pi}(\cdot,\psi)\psi}{\varphi(0)||\psi||\sqrt{|k|}} + \tilde{w}^{\frac{1}{2}}(k,0),\ k \in U_\delta^0(0), \tag{12.32}$$

bunda $\tilde{w}^{\frac{1}{2}}(k,0)$ *operator* $k \in U_\delta(0)$ *da uzluksiz*

Proof. $a)$ F.q. 0 nuqta regulyar nuqta bo'lsin.

$G(0,0)$ operator kompakt va 1 soni bu operator uchun xos qiymat bo'lmaganligi uchun etarlicha kichik $|k|, k \in U_\delta(0)$ lar uchun $(I - G(0,0))^{-1}-$ mavjud bo'ladi. (12.8) yoyilmaga ko'ra

$$||w(k,0)|| < C < \infty$$

ni hosil qilamiz.

$b)$. $h(0)$ operator nol energiyali resonansga ega bo'lsin. $\mathcal{H}_0$ orqali ψ ga mos qism fazoni belgilaymiz (bunda ψ funktsiya $G(0,0)\psi = \psi$ tenglama yechimi) va P_1 orqali uning ortogonal to'ldiruvchisi $\mathcal{H}_1$ ga mos proektorni belgilaymiz, ya'ni $P_0 \oplus P_1 = I$.

F.q. $A = A(k,0) = I - v^{\frac{1}{2}} r_0(k,0) v^{\frac{1}{2}}$ operator quyidagicha aniqlangan matritsaviy operator bo'lsin:

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix},$$

bunda $A_{ij} = P_i A P_j : \mathcal{H}_j \to \mathcal{H}_i,\ i,j = 0,1.$

$G(0,0)\psi = \psi$, $P_0(I - G(0,0))P_1 = 0$, bo'lganligi uchun 12.7 lemmaga ko'ra

$$A_{00} = |k|\big(-P_0 G_1 P_0 - |k| P_0 G_2(k)\big) P_0,$$
$$A_{01} = |k|\big(-P_0[G_1 + |k| G_2(k)] P_1\big),$$
$$A_{10} = A_{01}^*,$$
$$A_{11} = \big(P_1(I - G_0)P_1 - |k| P_1[G_1 + |k| G_2(k)] P_1\big). \tag{12.33}$$

It is more convenient instead of A to consider the operator

$$B = PAP, \quad P := \begin{pmatrix} \frac{P_0}{\sqrt{|k|}} & 0 \\ 0 & P_1 \end{pmatrix}.$$

U holda (12.33) ga ko'ra

$$B_{00} = -P_0 G_1 P_0 - |k| P_0 G_2(k) P_0,$$
$$B_{01} = -\sqrt{|k|} P_0 [G_1 + |k| G_2(k)] P_1,$$
$$B_{10} = B_{01}^*,$$
$$B_{11} = P_1(I - G(0,0))P_1 - |k| P_1 [G_1 + |k| G_2(k)] P_1.$$

Va demak $B = B^{(0)} + K$, bunda

$$B^{(0)} = \begin{pmatrix} P_0 G_1 P_0 & 0 \\ 0 & P_1(I - G(0,0))P_1 \end{pmatrix}$$

va $K = O(|k|),\ k \to 0.$

P_1 operatorning aniqlanishidan $F = (P_1(I - G(0,0))P_1)^{-1}$ operator $\mathcal{H}_1$ fazoda ta'sir qiladi.

$P_0 = ||\psi||^{-2}(\cdot,\psi)\psi$ tenglik va $(v^{\frac{1}{2}}, \psi) = \varphi(0)$ ga ko'ra (12.7) dan

$$-P_0 G_1 P_0 = \frac{1}{4\pi||\psi||^2} P_0 (\psi, v^{\frac{1}{2}})^2 = \frac{\varphi^2(0)}{8\pi||\psi||^2} P_0$$

ni hosil qilamiz.

O'z navbatida

$$(-P_0G_1P_0)^{-1} = \frac{8\pi||\psi||^2}{\varphi^2(0)}P_0 = \frac{4\pi}{\varphi^2(0)}(\cdot,\psi)\psi.$$

Bundan $B = (I + K(B^{(0)})^{-1})B^{(0)}$ va $K = O(|k|),\ k \to 0$ tengliklarga ko'ra

$$B^{-1} = (B^{(0)})^{-1} + O(|k|) = \begin{pmatrix} \frac{8\pi(\cdot,\psi)\psi}{\varphi^2(0)|k|} & 0 \\ 0 & F \end{pmatrix} + O(|k|)$$

ga ega bo'lamiz.

$w(k,0) = (A(k,0))^{-1} = PB^{-1}P$ tenglikni hisobga olsak (12.31) ning isbotiga kelamiz.

Endi (12.32)munosabat o'rinli ekanligini ko'rsatamiz.

$h(0) \geq 0$ ekanligidan $r(k,0) \geq 0$ ga egamiz, bundan esa $w(k,0) \geq I \geq 0$ ni hosil qilamiz.

Shuning uchun

$$\left(\frac{8\pi(\cdot,\psi)\psi}{\varphi^2(0)||\psi||^2|k|}\right)^{\frac{1}{2}} = \frac{2\sqrt{2\pi}(\cdot,\psi)\psi}{\varphi(0)||\psi||\sqrt{|k|}}$$

tenglik o'rinlidir.

A va B musbat (arbitrary) operatorlar uchun (**[?]** ga q.): $||B^{\frac{1}{2}} - A^{\frac{1}{2}}|| \leq ||B - A||^{\frac{1}{2}}$ tengsizlik bajarilishidan va (12.31) dan quyidagiga ega bo'lamiz:

$$\left|\left|(w(k,0))^{\frac{1}{2}} - \frac{2\sqrt{2\pi}(\cdot,\psi)\psi}{\varphi(0)||\psi||\sqrt{|k|}}\right|\right| \leq C.$$

□

13. Ikki zarrachali sistemaga mos Fridrixs modellari va ularning spekti haqida

Kirish

Mazkur bo'limda $\mathbb{Z}^3$ panjarada qo'zg'alishi bir o'lchamli ikki zarrachali sistemaga mos Fridrixs modellari oilasini qaraymiz. Ishning asosiy maqsadi Fridriechs modellari oilasining spektral xossalarini mukammal matematik o'rganish va ushbu oilaga mos Fredgolm determinanti uchun bo'saga energiyali yoyilmalarni olishdan iborat.

Muhokama uchun [3, 5, 6, 8, 33, 40, 55, **?**, **?**] larga va panjaradagi kvant zarrachali sistema spektri qo'zg'alishini o'rganish uchun [36, 49, 62] larga qarang). Modellarning bu turi kvant mexanikasi [30, 34], qattiq jismlar fizikasi [52, 47, **?**, 27] va panjaradagi maydonlar nazariyasi [46, 44, 45] da uchraydi.

Panjaradagi Shroedinger operatorlariga o'xshash va uzluksiz Shroedinger operatorlariga zid ravishda $h_\mu(p),\ p\in(-\pi,\pi]^3$, $\mu>0$ Fridrixs modellari oilasi parametrik tarzda ichki bog'lanish p, ga, panjara katagi bo'ylab o'zgaruvchi kvaziimpulsga bog'liq va shuning uchun o'sha panjaradagi Schrödinger operatorlariga o'xshash spektral xossalarga ega . Asl Fridrixs modeli va uning umumlashmarining spektri va rezolventalari [34, 30, 37, 61] ishlarda o'rganilgan va muhim spektr tubidan quyida yotgan xos qiymatlarining cheksizligi isbotlangan.

[44, 45] ishlarda Fridrixs modellarining maxsus oilasi qaralgan va sistema to'la kvazi-impulsi $p, p\in(-\pi,\pi]^d, d=1,2$ ning bazi xususiy qiymatlari atrofida yotuvchi qiymatlar uchun xos qiymatlarning paydo bo'lishi isbotlangan. [6] da ikki zarrachali Shroedinger operatorlarining katta bir sinfi uchun kvaziimpuls ($0\neq p\in\mathbb{T}^3$) ning barcha nolmas qiymatlarida ($h_\mu(0)$ yoki nol energiyali rezonans yoki nol xos qiymatga ega bo'lsa) $h_\mu(p), p\in(-\pi,\pi]^3$ ning xos qiymatlari mavjudligi isbotlangan.

Mazkur bo'limda ikkita asosiy natija isbotlangan.

Birinchi natija $h_\mu(p), p\in(-\pi,\pi]^3$ operator kvaziimpuls $0\neq p\in\mathbb{T}^3$ ning barcha no'lmas qiymatlarida yagona $e_\mu(p)$ xos qiymatga ega ekanligi hamda uning quyi va yuqori chegaralari mavjudligini bildiradi. Bu natija $e_\mu(p)$ xos qiymatning μ ga monoton bog'liqligini ta'kidlaydi.

Ikkinchi natija Fredgolm determinanti va Birman- Shvinger operatori uchun koordinatalar boshining kichik $\delta-$ atrofida $p\in(-\pi,\pi]^3$ kvaziimpulsning darajalari boyicha yoyilmasini ifodalaydi. Xususan bu yoyilmalar Fredgolm determinanti va Birman -Shvinger operatori uchun $w=(m-z)^{\frac12}\geq 0,\ z\leq m$ o'zgaruvchining funktsiyasi sifatida $h_\mu(p)$ operatorning muhim spektri tubigacha differensiallanuvchi davomga ega ekanligini isbotlaydi (Teorema 14.15).

Shuni qayd qilamizki, agar u va v funktsiyalar mos ravishda $(\mathbb{T}^3)^2$ va $\mathbb{T}^3$, da analitik bo'lsa, u holda Fredholm determinani va Birman-Schwinger operatori uchun aniq yoyilmalarni olish mumkin.

Bo'lim quyidagi tuzilmaga ega.

Ikkinchi bo'limda masalaning qo'yilishi va asosiy natijalarni bayon qilamiz.

Asosiy natijalar isboti to'rtinchi bo'limda bayon etilgan va ular uchinchi bo'limdagi qator lemmalarga asoslangan.

Ilovada Fridrixs modellari oilasining muhim bir sinfi uchun ikkinchi bo'limdagi barcha talablar bajarilishini ko'rsatamiz.

Mazkur ish mobaynida quyidagi shartli belgilardan foydalanamiz: $\mathbb{T}^3$ orqali uch o'lchamli tor belgilangan .

Har bir $\delta>0$ uchun $U_\delta(0)=\{p\in\mathbb{T}^3:|p|<\delta\}$ belgilash kordinatalar boshining yetarlicha kichik δ-atrofini aniqlaydi.

$L_2(\mathbb{T}^3)$ orqali $\mathbb{T}^3$ da aniqlangan kvadrati bilan integrallanuvchi funktsiyalarning Gilbert fazosini ifodalyadi.

$\mathcal{B}(\theta,U_\delta(0)),\ 1/2<\theta<1$ orqali $\overline{U_\delta(0)}$ dagi Hólder uzluksiz funktsiyalarining Banach fazosi belgilangan. Bunda $U_\delta(0)$ da aniqlangan θ ko'rsatkichli silliq funktsiyalar fazosining yopig'i

$$\|f\|_\theta=\sup_{\substack{t,\ell\in U_\delta(0)\\ t\neq\ell}}\Big[|f(t)|+|t-\ell|^{-\theta}|f(t)-f(\ell)|\Big]$$

normaga nisbatan olinadi.

$n-$ tartibgacha uzluksiz xususiy hosilalarga ega bo'lgan $f:\mathbb{T}^3\to\mathbb{R}$ funktsiyalar to'plami $C^{(n)}(\mathbb{T}^3)$ orqali belgilanadi. Xususan $C^{(0)}(\mathbb{T}^3)=C(\mathbb{T}^3)$.

14. $h_\mu(p)$ model operatori, asosiy shartlar (farazlar)va asosiy natijalarning bayoni

F.q $u-$ $(\mathbb{T}^3)^2$ torda aniqlangan haqiqiy qiymatli muhim chegaralangan funktsiya va $\varphi\in L_2(\mathbb{T}^3)$ haqiqiy qiymatli funktsiya, μ musbat haqiqiy son bo'lsin. Biz $L_2(\mathbb{T}^3)$ da aniqlangan chegaralangan o'z-o'ziga qo'shma $h_\mu(p),\ p\in\mathbb{T}^3$, operatorlar (Fridrixs modellari)ning quyidagi

$$h_\mu(p)=h_0(p)-\mu v, \tag{14.1}$$

oilasini qaraymiz, bunda

$$(h_0(p)f)(q) = u(q,p)f(q),\ \ f \in L_2(\mathbb{T}^3),$$

v esa integral operator

$$(vf)(q) = \varphi(q)\int_{\mathbb{T}^3}\varphi(t)f(t)dt,\ \ f \in L_2(\mathbb{T}^3).$$

Eslatma 14.1.

(14.2) $$u(p,q) = \varepsilon(p) + \varepsilon(p-q) + \varepsilon(q),$$

ko'rinishga ega bo'lgan u funktsiya uch o'lchamli $\mathbb{Z}^3$ panjarada ikki zarrachali (bozonlar) sistemasining $h_0(p)$ operatoriga mos keladi va bu operatorga erkin Hamiltonian deb ataladi.

Ushbu bo'lim davomida quyidagi qo'shimcha shartlarni faraz qilamiz.

Talab 14.2. *(i) u funktsiya $(\mathbb{T}^3)^2$ da (p,q) o'zgaruvchilarga nisbatan juft va $(0,0) \in (\mathbb{T}^3)^2$ nuqtada yagona aynimagan minimumga ega hamda bu funktsiyaning barcha uchinchi tartibli hosilalari $(\mathbb{T}^3)^2$ da uzluksiz va $\mathcal{B}(\theta,(U_\delta(0))^2)$ fazoga tegishli.*
(ii) Shunday musbat aniqlangan U matrisa va l, l_1, l_2 $(l_1, l_2 > 0, l \neq 0)$ haqiqiy sonlar mavjudki, quyidagi tengliklar o'rinli:

$$\left(\frac{\partial^2 u(0,0)}{\partial p^{(i)}\partial p^{(j)}}\right)_{i,j=1}^3 = l_1 U,\ \left(\frac{\partial^2 u(0,0)}{\partial p^{(i)}\partial q^{(j)}}\right)_{i,j=1}^3 = lU,\ \left(\frac{\partial^2 u(0,0)}{\partial q^{(i)}\partial q^{(j)}}\right)_{i,j=1}^3 = l_2 U$$

.

Eslatma 14.3. *u funktsiya $\mathbb{T}^3$ da juft hamda yagona aynimagan minimumga ega, shuninig uchun umumiylikka ziyon etkazmasdan u funktsiya $(0,0) \in (\mathbb{T}^3)^2$ nuqtada yagona aynimagan minimumga ega deb olamiz.*

Eslatma 14.4. *Ko'rinib turibdiki, 14.2 talab $l_1 l_2 > l^2$ tengsizlikning o'rinli ekanini anglatadi.*

Talab 14.5. *φ uzluksiz funktsiya $\mathbb{T}^3$ da yo juft yo toq va $\varphi \in \mathcal{B}(\theta, U_\delta(0))$.*

Quyidagicha

$$u_p(q) = u(p,q), \qquad m = \min_{p,q\in\mathbb{T}^3} u(p,q),$$

$$u_{\min}(p) = \min_{q\in\mathbb{T}^3} u_p(q), \quad u_{\max}(p) = \max_{q\in\mathbb{T}^3} u_p(q)$$

va

(14.3) $$\Lambda(p,z) = \int\limits_{\mathbb{T}^3} \frac{\varphi^2(t)dt}{u_p(t)-z}, \quad p \in \mathbb{T}^3,\ z \notin \mathbb{C}\setminus[u_{\min}(p), u_{\max}(p)].$$

belgilashlar kiritamiz.

Eslatma 14.6. *14.2 talabning (i) qismi va Lebegning yaqinlashish haqidagi teoremasiga ko'ra, ixtiyoriy $z < m$ uchun $\Lambda(\cdot, z)$ funktsiyaning barcha ikkinchi tartibli hosilalari $C^{(2)}(\mathbb{T}^3)$ fazoga qarashli (yoki $C^{(2)}(\mathbb{T}^3)$ fazoning elementi) bo'ladi.*

Ma'lumki $h_0(p)-$ ko'paytirish operatori, v esa qo'zg'alishi bir o'lchamli operator (demak kompakt) va shuning uchun Veyl teoremasiga ko'ra $h_\mu(p)$ operatorning muhim srektri haqiqiy o'qning quyidagi

$$\sigma_{ess}(h_\mu(p)) = [u_{\min}(p), u_{\max}(p)]$$

intervalidan iborat.

Eslatma 14.7. *Eslatamizki, ba'zi $p \in \mathbb{T}^3$ larda $h_\mu(p)$ operatorning muhim spektri yagona aynigan nuqtani saqlovchi $[u_{\min}(p), u_{\min}(p)]$ to'plamdan iborat bo'lishi mumkin. Shuning uchun biz $h_\mu(p)$ operator muhim spektrning barcha $p \in \mathbb{T}^3$ lar uchun absolyut uzluksizligini tasdiqlay (ta'kidlay)olmaymiz.*

Masalan, 14.2 ko'rinishdagi funktsiya va $p = (\pi,\pi,\pi) \in \mathbb{T}^3$, uchun

(14.4) $$\varepsilon(q) = 3 - \cos q_1 - \cos q_2 - \cos q_3, \quad q = (q_1, q_2, q_3) \in \mathbb{T}^3$$

ni qarash yetarli.

Ta'rif 14.8. *14.2 talabning (i) bandi o'rinli bo'lsin.*

Agar 1 *soni*

$$(\mathrm{G}\psi)(q) = \mu\varphi(q) \int\limits_{\mathbb{T}^3} \frac{\varphi(t)\psi(t)dt}{u_0(t)-m}, \quad \varphi \in C^{(0)}(\mathbb{T}^3)$$

operator uchun xos qiymat bo'lsa, va unga mos ψ xos funktsiya (o'zgarmasga ko'paytirish aniqligida) $\psi(0) \neq 0$ shartni qanoatlantirsa, $h_\mu(0)$ operator bo'sag'a energiya rezonansiga ega deyiladi.

Eslatma 14.9. *F.q. 14.2 talabning (i) bandi va 14.5 talab o'rinli bo'lsin.*

(i) *Agar $\varphi(0) \neq 0$ va $h_\mu(0)$ operator bo'sag'a energiya rezonansiga ega bo'lsa, u holda*

$$f(q) = \frac{\varphi(q)}{u_0(q)-m} \tag{14.5}$$

funktsiya $h_\mu(0)f = mf$ tenglamani qanoatlantiradi va $f \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$ bo'ladi (Lemma15.2 ga qar.).
(ii) *Agar $\varphi(0) = 0$ va $z = m$ bo'sag'a $h_\mu(0)$ operatorning xos qiymati bo'lsa, u holda* (14.5) *kabi aniqlangan f funktsiya $h_\mu(0)f = mf$ tenglamani qanoatlantiradi va $f \in L_2(\mathbb{T}^3)$ bo'ladi (Lemma 15.3 ga qar.).*

Faraz qilaylik

$$\mu_0 = \Lambda^{-1}(0, m)$$

bo'lsin.

Eslatma 14.10. *Shuni esda tutingki, $\mu = \mu_0$ va $\varphi(0) \neq 0$ ($\mu = \mu_0$ va $\varphi(0) = 0$) shartlar $h_\mu(0)$ operator bo'sag'a energiya rezonansiga (Lemma 15.2ga qarang) (mos ravishda $h_\mu(0)$ operator bo'sag'a xos qiymatga (Lemma 15.3) ga qarang) ega ekanligini anglatadi.*

Eslatma 14.11. *$h_{\mu_0}(0)$ operator muhim spektri $\sigma_{ess}(h_{\mu_0}(0))$ ning $z = m$ tubi $h_{\mu_0}(0)$ operator uchun yo bo'sag'a energiyali rezonons yo xos qiymat bo'ladi.*

Qaralayotgan $h_\mu(p)$ operatorning spektral xossalarini to'la (aniq) o'rganish uchun quyidagicha faraz qilamiz.

Talab 14.12. (i) *$\Lambda(\cdot, m(\cdot))$ funktsiya koordinatalar boshida yagona n minimumga ega bo'lsin, ya'ni barcha $0 \neq p \in \mathbb{T}^3$ lar uchun quyidagi*

$$\Lambda(p, u_{\min}(p)) - \Lambda(0, u_{\min}(0)) > 0 \tag{14.6}$$

tengsizlik o'rinli bo'lsin.

(ii) *$\Lambda(\cdot, m)$ funktsiya koordinatalar boshida yagona maksimumga ega bo'lsin va birorta $c > 0$ uchun quyidagi*

$$\Lambda(0, m) - \Lambda(p, m) > c|p|^2, \quad 0 \neq p \in U_\delta(0)$$

tengsizlik bajarilsin.

Eslatma 14.13. *Agar barcha $0 \neq p \in \mathbb{T}^3$ va deyarli barcha $q \in \mathbb{T}^3$ lar uchun*

$$u_p(q) - u_{\min}(p) < u_0(q) - u_{\min}(0)$$

tengsizlik o'rinli bo'lsa, (14.12) *talabning (i) bandi shubhasiz bajariladi. Ilovada* (14.2) *ko'rinishdagi funktsiyalar uchun* (14.12) *talabning bajarilishini ko'rsatamiz.*

Quyidagi teorema panjaradagi ikki zarrachali Hamiltonian uchun xarakterli natijani ifodalaydi ([6] ga qarang).

Teorema 14.14. *F. q. 14.2, 14.5 va 14.12 talablar o'rinli bo'lsin. U holda barcha $p \in \mathbb{T}^3 \setminus \{0\}$ lar uchun $h_{\mu_0}(p)$ operator yagona $e_{\mu_0}(p)$ xos qiymatga ega bo'ladi va*

$$m < e_{\mu_0}(p) < u_{\min}(p),\ 0 \neq p \in \mathbb{T}^3.$$

munosabat o'rinli.

(ii) *$\mu > \mu_0$ uchun $h_\mu(p)$, $p \in \mathbb{T}^3$ operator yagona $e_\mu(p)$ xos qiymatga ega. Shuningdek*

$$e_\mu(p) < e_{\mu_0}(p) < u_{\min}(p),\ 0 \neq p \in \mathbb{T}^3$$

va

$$e_\mu(0) < m$$

munosabatlar o'rinli.

F.q. $\mathbb{C}$ kompleks sonlar maydoni bo'lsin. Ixtiyoriy $p \in \mathbb{T}^3$ uchun $\mathbb{C}\backslash[u_{\min}(p), u_{\max}(p)]$ da aniqlangan $\Delta_\mu(p,\cdot)$ funktsiya ($h_\mu(p)$) operatorga mos Fredgolm determinanti) ni kiritamiz:

$$\Delta_\mu(p,z) = 1 - \mu \int\limits_{\mathbb{T}^3} (u(p,t)-z)^{-1}\varphi^2(t)dt.$$

Endi biz $\mathbb{Z}^3$ panjaradagi uch zarrachali sistemaga mos model operatorning spektrini tahlil qilishda muhim bo'lgan Fredgolm determinanti yoyilmasi tug'risidagi natijani keltiramiz:

Teorema 14.15. *F.q. 14.2 va 14.5 talablar o'rinli bo'lsin.*
(i) *Ixtiyoriy* $z < u_{\min}(p)$ *uchun* $\Delta_\mu(\cdot,z)$ *funktsiya* $C^{(2)}(\mathbb{T}^3)$ *sinfga qarashli va quyidagi*

$$\Delta_\mu(p,z) = \Delta_\mu(0,z) + \Delta_\mu^{res}(p,z) \quad p \in U_\delta(0) \tag{14.7}$$

yoyilma o'rinli, bunda $p \to 0,\ z \le u_{\min}(p)$ *da* $\Delta_\mu^{res}(p,z) = O(p^2)$ *ga tekis yaqinlashadi.*
(ii) $\Delta_\mu(\cdot,z)$ *uchun quyidagi*

$$\Delta_\mu(p,z) = \Delta_\mu(0,0) + \frac{2\sqrt{2}\pi^2\mu\varphi^2(0)}{l_1^{\frac{3}{2}}\det(U)^{\frac{1}{2}}}\sqrt{m-z} + \Delta_\mu^{res}(z) \tag{14.8}$$

yoyilma o'rinli, bunda $z \to m-$ *da* $\Delta_\mu^{res}(m-z) = O(m-z)$ *va* $z < m$ *uchun* $(m-z)^{\frac{1}{2}} > 0.$

Eslatma 14.16. $\varphi(\cdot) \equiv const$ *va* $u(\cdot,\cdot)$ *funktsiya* (14.2), (14.4) *ko'rinishda bo'lgan hol uchun ushbu natijaning o'xshashi* [**?**] *ishda isbotlangan.*

Natija 14.17. (i) $h_{\mu_0}(0)$ *operator bo'sag'a energiya rezonansiga ega bo'lsin. U holda barcha* $p \in U_\delta(0)$ *va* $z \le m$ *uchun quyidagi yoyilma o'rinli.*

$$\Delta_{\mu_0}(p,z) = \frac{4\sqrt{2}\pi^2\mu_0\varphi^2(0)}{l_1^{\frac{3}{2}}\det(U)^{\frac{1}{2}}}\sqrt{u_{min}(p)-z} + \Delta_{\mu_0}^{res}(u_{min}(p)-z) + \Delta_{\mu_0}^{res}(p,z).$$

(ii) $z = m$ *bo'sag'a* $h_{\mu_0}(0)$ *operatorning xos qiymati bo'lsin. U holda barcha* $p \in U_\delta(0)$ *va* $z \le m$ *uchun quyidagi yoyilma o'rinli.*

$$\Delta_{\mu_0}(p,z) = \Delta_{\mu_0}^{res}(u_{min}(p)-z) + \Delta_{\mu_0}^{res}(p,z).$$

Eslatma 14.18. *14.17 natija Fredgolm determinanti uchun mos ravishda bo'sag'a energiya rezonansi va bo'sag'a xos qiymatining turli atroflaridagi bo'sag'a energiya yoyilmalarini beradi.*

Quyidagi 14.19 natija (mos holda 14.20 natija) $\mathbb{Z}^3$ panjaradagi uch zarrachali sistemaga mos model operatori uchun muhim spektr tubidan quyida joylashgan (yotgan) xos qiymatlari sonining cheksizligini (mos ravishda chekliligini) isbotlashda muhim rol o'ynaydi.

Natija 14.19. *F. q. 14.2 va 14.5 talablar bajarilsin. Bundan tashqari* $h_{\mu_0}(0)$ *operator bo'sag'a rezonansiga ega bo'lsin. U holda shunday* $c_1, c_2 > 0$ *sonlar mavjud bo'lib, quyidagi*

$$c_1|p| \le \Delta_{\mu_0}(p,m) \le c_2|p|, \quad p \in U_\delta(0), \tag{14.9}$$

$$\Delta_{\mu_0}(p,m) \ge c_1, \quad p \in \mathbb{T}^3 \setminus U_\delta(0) \tag{14.10}$$

tengsizliklar bajariladi.

Natija 14.20. *F.q. 14.2, 14.5 va14.12 talablar o'rinli bo'lsin.* $z = m$ *nuqta* $h_{\mu_0}(0)$ *operatorning xos qiymati bo'lsin. U holda biror* $c > 0$ *uchun quyidagi*

$$\Delta_{\mu_0}(p,m) \ge cp^2, \quad p \in U_\delta(0)$$

tengsizlik o'rinli.

15. $h_\mu(p)$ OPERATORNING SPEKTRAL XOSSALARI.

Bu bo'limda biz $h_\mu(p)$, $p \in \mathbb{T}^3$, operatorning spektral xossalarini bo'sag'a energiya rezonansi va bo'sag'a xos qiymatlariga urg'u bergan holda (asosiy etiborni qaratgan holda) o'rganamiz.

$r_0(p,z) = (h_0(p) - zI)^{-1}$ orqali $h_0(p)$ operatorning rezolbentasini, ya'ni

$$(u_p(\cdot) - z)^{-1}, \quad z \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$$

funktsiyaga ko'paytirish operatorini belgilaymiz, bunda I - $L_2(\mathbb{T}^3)$ dagi birlik operator. $u_p(\cdot)$, $\quad p \in U_\delta(0)$ funktsiya $q = q_0(p) \in U_\delta(0)$ da yagona aynimagan minimumga ega (Lemma (15.4) ga qarang) va shuning uchun Lebegning yaqinlashish haqidagi teoremasiga ko'ra quyidagi

$$\Delta_\mu(p,m) = \lim_{z \to m-0} \Delta_\mu(p,z), \quad p \in U_\delta(0)$$

chekli limit mavjud.

Lemma 15.1. *Faraz qilaylik 14.2 ning (i) bandi va $\varphi \in C^{(0)}(\mathbb{T}^3)$ talablar o'rinli bo'lsin. Ixtiyoriy $\mu > 0$ va $p \in \mathbb{T}^3$ uchun quyidagi tasdiqlar ekvivalent:*
(i) $h_\mu(p)$ *operator muhim spektr tubidan quyida* $z \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$ *xos qiymatga ega;*
(ii) $\Delta_\mu(p,z) = 0$, $z \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$;
(iii) *biror bir* $z' \leq u_{\min}(p)$ *uchun* $\Delta_\mu(p,z') < 0$.

Proof. $(i) \to (ii) \to (iii) \to (ii)$ ekanligini isbotlaymiz. v operatorning musbatligidan, uning musbat kvadrat ildizi mavjudligi kelib chiqadi va biz uni $v^{\frac{1}{2}}$ orqali belgilaymiz.

Ixtiyoriy $\mu > 0$ va $p \in \mathbb{T}^3$ uchun $z \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$ son faqat va faqat $\lambda = 1$ soni

$$G_\mu(p,z) = \mu v^{\frac{1}{2}} r_0(p,z) v^{\frac{1}{2}}$$

operatorning xos qiymati bo'lgandagina $h_\mu(p)$ operatorning xos qiymati bo'ladi. (Bu Birman-Schwinger prinsipiga asosan kelib chiqadi). Ushbu $v^{\frac{1}{2}}$ operator

$$(v^{\frac{1}{2}} f)(q) = ||\varphi||^{-1} \varphi(q) \int\limits_{\mathbb{T}^3} \varphi(t) f(t) dt, \quad f \in L_2(\mathbb{T}^3)$$

ko'rinishda bo'lganligi uchun $G_\mu(p,z)$ operator

$$(G_\mu(p,z) f)(q) = \frac{\mu \Lambda(p,z)}{||\varphi||^2} \varphi(q) \int\limits_{\mathbb{T}^3} \varphi(t) f(t) dt, \quad f \in L_2(\mathbb{T}^3),$$

ko'rinishda ifodalanadi, bunda $\Lambda(p,z)$ funktsiya (14.3) orqali aniqlangan.

Fredhol'm teoremasiga ko'ra, $\lambda = 1$ soni faqat va faqat $\Delta_\mu(p,z) = 0$ bo'lgandagina $G_\mu(p,z)$ operatorning xos qiymati bo'ladi. Demak $(i) \Leftrightarrow (ii)$ isbotlandi.

Endi $(ii) \Leftrightarrow (iii)$ munosabatni isbotlaymiz. Biror bir $z_0 \in \mathbb{C} \setminus [u_{\min}(p), u_{\max}(p)]$ uchun $\Delta_\mu(p,z_0) = 0$ bo'lsin. $h_\mu(p)$ operator o'z-o'ziga qo'shma va shuning uchun $(i) \Leftrightarrow (ii)$ ga ko'ra z_0 soni haqiqiy degan xulosaga kelamiz. Barcha $z > u_{\max}(p)$ lar uchun $\Delta_\mu(p,z) > 1$ ga egamiz va shuning uchun $z \in (-\infty, u_{\min}(p))$. Ixtiyoriy $p \in \mathbb{T}^3$ uchun $\Delta_\mu(p,\cdot)$ funktsiya $z \in (-\infty, u_{\min}(p))$ bo'yicha kamayuvchi bo'lganligi uchun biror bir $z_0 < z' \leq u_{\min}(p)$ da $\Delta_\mu(p,z') < \Delta_\mu(p,z_0) = 0$ tengsizlikka ega bo'lamiz va demak $(ii) \Leftrightarrow (iii)$ isbotlandi.

$(iii) \Leftrightarrow (ii)$. Biror $z' \leq u_{\min}(p)$ uchun $\Delta_\mu(p,z') < 0$ o'rinli bo'lsin. Ixtiyoriy $p \in \mathbb{T}^3$ uchun $\lim\limits_{z \to -\infty} \Delta_\mu(p,z) = 1$ ga egamiz, $\Delta_\mu(p,\cdot)$ funktsiya $z \in (-\infty, u_{\min}(p))$ bo'yicha uzluksiz va shuning uchun shunday $z_0 \in (-\infty, z')$ mavjudki $\Delta_\mu(p,z_0) = 0$ tenglik o'rinli bo'ladi. Bu esa isbotni tugallaydi. □

Quyidagi lemmalar $h_{\mu_0}(0)$ operator muhim spektri tubi bo'sag'a energiya rezonansi yoki bo'sag'a xos qiymati bo'lish- bo'lmasligini ifodalaydi.

Lemma 15.2. *F. q. 14.2 talabning (i) bandi bajarilsin va $\varphi \in C^0(\mathbb{T}^3)$ bo'lsin.*

Quyidagi tasdiqlar ekvivalent:
(i) $h_\mu(0)$ *operator bo'sag'a energiya rezonansiga ega va*

$$\varphi(q)(u_0(q) - m)^{-1} \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3). \tag{15.1}$$

(ii) $\varphi(0) \neq 0$ *va* $\Delta_\mu(0,m) = 0$.
(iii) $\varphi(0) \neq 0$ *va* $\mu = \mu_0$.

Proof. $(i) \to (ii) \to (iii) \to (i)$ ekanligini isbotlaymiz. $h_\mu(0)$ operator biror $\mu > 0$ uchun bo'sag'a energiya rezonansiga ega bo'lsin. U holda 14.8 ta'rifga ko'ra

$$\psi(q) = (G\psi)(q), \quad \psi \in C(\mathbb{T}^3) \tag{15.2}$$

tenglama $C(\mathbb{T}^3)$ da $\psi(0) \neq 0$ ni qanoatlantiruvchi oddiy yechimga ega. Bu yechim (o'zgarmas koeffitsient farqi bilan) φ funktsiyaga teng. Shuning uchun $\Delta_\mu(0, m) = 0$ va shunday qilib, $\mu = \mu_0$. Faraz qilaylik $\varphi(0) \neq 0$ va $\mu = \mu_0$ bo'lsin. U holda $\Delta_\mu(0, m) = 0$ tenglik o'rinli bo'ladi. Bundan φ funktsiya (15.2) tenglamaning yechimi, yani $h_\mu(0)$ operator bo'sag'a energiya rezonansiga ega. Shunday qilib, $u_0(\cdot)$ funktsiya $p = 0 \in \mathbb{T}^3$ nuqtada yagona aynimagan minimumga ega va $\varphi(0) \neq 0$ (15.1) munosabat o'rinli. □

Lemma 15.3. *Faraz qilaylik 14.2 talabning ning (i) bandi o'rinli va $\varphi \in C^0(\mathbb{T}^3)$ bo'lsin.*
Quyidagi tasdiqlar ekvivalent
(i) *$h_\mu(0)$ operator bo'sag'a xos qiymatga ega;*
(ii) *$\varphi(0) = 0$ va $\Delta_\mu(0, m) = 0$;*
(iii) *$\varphi(0) = 0$ va $\mu = \mu_0$.*

Proof. Ushbu $(i) \to (ii) \to (iii) \to (i)$ munosabatni isbotlaymiz.
F. q. $f \in L_2(\mathbb{T}^3)$ funktsiya $h_\mu(0)$ operatorning m xos qiymatiga mos xos funktsiyasi bo'lsin. U holda

$$(u_0(q) - m)f(q) - \mu\varphi(q) \int\limits_{\mathbb{T}^3} \varphi(t)f(t)dt = 0 \quad \text{va} \quad \int\limits_{\mathbb{T}^3} \varphi(t)f(t)dt \neq 0 \tag{15.3}$$

tenglik o'rinli bo'ladi. Bundan esa $\varphi(0) = 0$ ekanligi kelib chiqadi. Demak (Ixtiyoriy o'zgarmasga ko'paytirish aniqligida) f ning

$$f(q) = (u_0(q) - m)^{-1}\varphi(q) \tag{15.4}$$

ko'rinishda bo'lishini topdik.

Shunday qilib, (15.3) dan $\Delta_\mu(0, m) = 0$ va $\mu = \mu_0$ kelib chiqadi. Endi $\varphi(0) = 0$ va $\mu = \mu_0$ bo'lsin. U holda $\Delta_\mu(0, m) = 0$ va (15.4) kabi aniqlangan f funktsiya $h_\mu(0)f = mf$ tenglamani qanoatlantiradi. $u_0(\cdot)$ funktsiya $p = 0 \in \mathbb{T}^3$ da yagona aynimagan minimumga ega bo'lgani va $\varphi(0) = 0$ bo'lgani uchun $f \in L_2(\mathbb{T}^3)$ ga kelamiz. □

Lemma 15.4. *14.2 talab o'rinli bo'lsin. U holda*
(i) $u_{min}(p) = \min_{q\in\mathbb{T}^3} u(p, q)$ *funktsiya juft va uning barcha ikkinchi tartibli xususiy hosilalari $B(\theta, \mathbb{T}^3)$ ga qarashli (tegishli) bo'ladi va*

$$p \to 0 \quad \textit{da} \quad u_{min}(p) = m + \frac{l_1^2 - l_2^2}{2l_1}(Up, p) + O(|p|^{2+\theta}) \tag{15.5}$$

asimtotika o'rinli.
(ii) *Biror $p \in \mathbb{T}^3$ uchun $q_0(p)$ nuqta $u_p(p)$ ning minimumi bo'lsin, ya'ni $u_p(q_0(p)) = \min_{q\in\mathbb{T}^3} u_p(q)$.*
U holda $q_0(-p) = -q_0(p)$ va $u_p(q_0(p)) = \min_{q\in\mathbb{T}^3} u_p(q) = u_p(-q_0(p))$ tengliklar o'rinli.
(iii) *Ixtiyoriy $p \in U_\delta(0)$ uchun $q \in \mathbb{T}^3$ nuqta $u_p(\cdot)$ ning yagona aynimagan minimumi bo'ladi va*

$$p \to 0 \quad \textit{da} \quad q_0(p) = -\frac{l_2}{l_1}p + O(|p|^{2+\theta}). \tag{15.6}$$

Proof. (i) va (ii) larning ekvivalentligini isbotlaymiz. $u(p, q)$ funktsiya juft bo'lgani uhun

$$u_{min}(-p) = \min_{q\in\mathbb{T}^3} u_{-p}(q) = \min_{q\in\mathbb{T}^3} u_p(-q) = \min_{-q\in\mathbb{T}^3} u_p(q) =$$

$$\min_{q\in\mathbb{T}^3} u_p(q) = u_{min}(p), p \in \mathbb{T}^3$$

$$u_p(-q_0(p)) = u_{-p}(q_0(p)) = \min_{q\in\mathbb{T}^3} u_{-p}(q) = \min_{q\in\mathbb{T}^3} u_p(-q) = \min_{-q\in\mathbb{T}^3} u_p(q) =$$

$$\min_{q\in\mathbb{T}^3} u_p(q) = u_p(q_0(p)).$$

$$u_p(q_0(-p)) = u_{-p}(-q_0(-p)) = \min_{-q\in\mathbb{T}^3} u_{-p}(-q) =$$

$$= \min_{-q\in\mathbb{T}^3} u_p(q) == \min_{q\in\mathbb{T}^3} u_p(-q) = u_p(-q_0(p))$$

tengliklar o'rinli bo'ladi.

14.2 talabning (i) qismidan $u_{min}(\cdot)$ ning barcha ikkinchi tartibli xususiy hosilalari $B(\theta, U_\delta(0))$ ga qarashli bo'lishi kelib chiqadi.

14.2 talabning (i) bandi va oshkormas funktsiya haqidagi teoremaga ko'ra $q_0(\cdot)$ ning barcha ikkinchi tartibli xususiy hosilalari $B(\theta, U_\delta(0))$ fazoga qarashli va u $u_p(\cdot)$ ning yagona aynimagan minimum nuqtasi bo'ladi. ([**?**] dagi lemma 3 ga qarang).

Teylor formulasidan (15.5) va (15.6) asimtotik formulalarni keltirib chiqarish mumkin.

□

$$\mathbb{C}_+ = \{z \in \mathbb{C} : Rez > 0\}, \quad \mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}, \quad \mathbb{R}_+^0 = \mathbb{R}_+ \cup \{0\}$$

va $u(\cdot,\cdot)$ funktsiya $U_\delta(0) \times \mathbb{T}^3$ da

$$\tilde{u}(p,q) = u_p(q + q_0(p)) - u_{\min}(p)$$

kabi aniqlangan bo'lsin.

Ixtiyoriy $p \in \mathbb{T}^3$ uchun $\mathbb{C}_+$ da $D(p,\cdot)$ analitik funktsiyani quyidagicha aniqlaymiz

$$D(p,w) = \int\limits_{\mathbb{T}^3} \frac{\varphi^2(q + q_0(p))dq}{\tilde{u}(p,q) + w^2}.$$

Lemma 15.5. *14.2 va 14.5 talablar o'rinli bo'lsin. U holda ixtiyoriy $w \in \mathbb{R}_+^0$ uchun $D(\cdot,w)$ funktsiya $C^{(2)}(U_\delta(0))$ sinfga tegishli, $D(\cdot,w)$ ning $w=0$ dagi o'ng hosilasi mavjud va quyidagi*

$$D(p,w) = D(0,0) - \frac{4\sqrt{2}\pi^2\varphi^2(0)}{l_1^{\frac{3}{2}}(detU)^{\frac{1}{2}}}w + D^{res}(w) + D^{res}(p,w)$$

yoyilma o'rinli, bunda $w \to +0$ da $D^{res}(w) = O(w^{1+\theta})$ tekis yaqinlashadi va $p \to 0$ da $D^{res}(p,w) = O(p^2)$, $w \in \mathbb{R}_+^0$ da tekis yaqinlashadi.

Proof. (i) Ixtiyoriy $p \in U_\delta(0)$ uchun $q_0(p)$ nuqta $u_p(\cdot)$ funktsiyaning aynimagan minimumi bo'ladi (15.4 Lemmaga q.) va $q_0(\cdot) \in C^{(2)}(U_\delta(0))$. $m(\cdot) \in C^{(2)}(U_\delta(0))$ bo'lgani uchun $D(\cdot,\cdot)$ ning aniqlanishi hamda 14.2 va 14.5 talablarga ko'ra $D(\cdot,w)$ funktsiyaning $C^{(2)}(U_\delta(0))$ sinfga qarashli ekanligini hosil qilamiz, bunda $C^{(n)}(U_\delta(0))$ ni xuddi $C^{(n)}(\mathbb{T}^3)$ kabi aniqlash mumkin.

Osongina ko'rish mumkinki, $|p|,|q| \to 0$ da

$$u_p(q + q_0(p)) - u_{\min}(p) = \frac{l_1}{2}(Uq,q) + o(|p||q|^2) + o(|q|^2)$$

Demak, qandaydir $C > 0$ hamda barcha $w \in \mathbb{R}_+^0$ va $i,j = 1,2,3$ uchun

$$\Big|\frac{\partial^2}{\partial p_i \partial p_j}\frac{1}{\tilde{u}(p,q) - w}\Big| \le C|q|^{-2}, \quad p,q \in U_\delta(0) \tag{15.7}$$

va

$$\Big|\frac{\partial^2}{\partial p_i \partial p_j}\frac{1}{\tilde{u}(p,q) - w}\Big| \le C, \quad p \in U_\delta(0), q \in \mathbb{T}^3 \setminus U_\delta(0) \tag{15.8}$$

tengsizliklar o'rinli.

Lebegning yaqinlashish haqidagi teoremasiga asosan

$$\frac{\partial^2}{\partial p_i \partial p_j}D(p,0) = \lim_{w \to 0+}\frac{\partial^2}{\partial p_i \partial p_j}D(p,w), p \in U_\delta(0).$$

Adamar lemmasini takroriy qo'llab, ([**?**] V.1, 512b. ga qar.)

$$D(p,w) = D(0,w) + \sum_{i=1}^{3}\frac{\partial}{\partial p_i}D(0,w)p_i + \sum_{i,j=1}^{3}H_{ij}(p,w)p_ip_j,$$

ni hosil qilamiz. Bunda ixtiyoriy $w \in \mathbb{R}_+^0$ uchun $H_{ij}(\cdot,w)$, $i,j = 1,2,3$ funktsiyalar $U_\delta(0)$ da uzluksiz va

$$H_{ij}(p,w) = \frac{1}{2}\int_0^1\int_0^1 \frac{\partial^2}{\partial p_i \partial p_j}D(x_1x_2p,w)dx_1dx_2.$$

(15.7) va (15.8) baholashlar ixtiyoriy $p \in U_\delta(0)$ uchun $w \in \mathbb{R}_+^0$ da

$$|H_{i,j}(p,w)| \le \frac{1}{2}\int_0^1\int_0^1 |\frac{\partial^2}{\partial p_i \partial p_j}D(x_1x_2p,w)|dx_1dx_2 \le C\big(1 + \int\limits_{U_\delta(0)} q^{-2}\varphi^2(q + q(p))dq\big)$$

tekis yaqinlashishini beradi.

Ixtiyoriy $w \in \mathbb{R}^0_+$ uchun $D(\cdot, w)$ funktsiya $U_\delta(0)$ da juft bo'lganligi uchun

$$\frac{\partial}{\partial p_i} D(p,w)\Big|_{p=0} = 0, \quad i = 1,2,3$$

ga ega bo'lamiz.

$ii)$ Endi biz $D(0,\cdot)$ funktsiyaning $w=0$ da o'ng hosilasi mavjudligini, ya'ni qandaydir $C>0$ uchun quyidagi

$$\lim_{w\to 0+} w^{-1}(D^{(1)}(w) - D^{(1)}(0)) = \frac{2\sqrt{2}\pi^2\,\mu\varphi^2(0)}{l_1^{\frac{3}{2}}(detU)^{\frac{1}{2}}}, \tag{15.9}$$

$$\Big|\frac{\partial}{\partial w} D(0,w) - \frac{\partial}{\partial w} D(0,0)\Big| < Cw^\theta, \quad w \in \mathbb{R}^0_+ \tag{15.10}$$

munosabat o'rinli ekanligini ko'rsatamiz.

$$D_1(\zeta) - D_1(0) = -\frac{1}{2}\int\limits_{U_\delta(0)} \zeta^2[(u_0(0,q)+\zeta^2)u_0(0,q)]^{-1}\varphi^2(q)dq \tag{15.11}$$

ni qaraymiz.

$u_0(0,\cdot)$ funktsiya $q=0$ da yagona aynimagan minimumga ega. Demak, Mors lemmasiga ko'ra ([32] ga qar.) markazi $t=0$ da va radiusi $\gamma>0$ bo'lgan aniq $w_\gamma(0)$ shartni $q=0$ nuqtaning $\tilde{W}(0)$ atrofiga akslantiruvchi shunday o'zaro bir qiymatli $q=\psi(t)$ akslantirish mavjudki,

$$w_0(0,\psi(t)) = t^2 \tag{15.12}$$

bajariladi, bunda $\psi(0)=0$ va $q=\psi(t)$ akslantirishning $J_\psi(t) \in \mathcal{B}(\theta, U_\delta(0))$ Yakobiani uchun

$$J_\psi(0) = \sqrt{2}l_1^{-\frac{3}{2}}(detU)^{-\frac{1}{2}}$$

tenglik o'rinli.

(15.11) integralda $q=\psi(t)$ o'zgaruvchi almashtirib va (15.12) tenglikni qo'llab,

$$D_1(\zeta) - D_1(0) = -\frac{\zeta^2}{2}\int\limits_{W_\gamma(0)} \frac{\varphi^2(\psi(t))J_\psi(t)}{t^2(t^2+\zeta^2)}dt \tag{15.13}$$

ni olamiz.

(15.13) integralda $t = r\omega$ sferik koordinatlarga o'tib, biz uni

$$D_1(\zeta) - D_1(0) = -\frac{\zeta^2}{2}\int_0^\gamma \frac{F(r)}{r^2+\zeta^2}dr \tag{15.14}$$

ko'rinishga keltiamiz, bunda

$$F(r) = \int_{\mathbb{S}^2} \varphi^2(\psi(r\omega))J_\psi(r\omega)d\omega,$$

$\mathbb{S}^2 -$ $\mathbb{R}^3$ dagi birlik sfera va $d\omega$ shu fazodagi birlik sfera elementi.

Endi φ, $J_\psi \in \mathcal{B}(\theta, U_\delta(0))$ ekanligidan foydalanib ko'rishimiz mumkinki,

$$|F(r) - F(0)| \le Cr^\theta, \; r \in [0,\delta]. \tag{15.15}$$

(15.15) tengsizlikni qo'llab,

$$\lim_{\zeta\to 0+} \frac{D_1(\zeta) - D_1(0)}{\zeta} = 2\sqrt{2}\pi l_1^{-\frac{3}{2}}\varphi^2(0)(detU)^{-\frac{1}{2}} \tag{15.16}$$

tenglikni osongina ko'rishimiz mumkin.

Shuning uchun $D_1(\cdot)$ funktsiya $\zeta=0$ da o'ng hosilaga ega va (15.9) tenglik o'rinli. □

16. ASOSIY NATIJALARNING ISBOTI

Teorema 14.14 ning isboti. (i) 14.12 talabning (i) va (ii) qismlaridan mos ravishda

$$\Delta_{\mu_0}(p, u_{\min}(p)) < \Delta_{\mu_0}(0, m) = 0,\ 0 \neq p \in \mathbb{T}^3$$

va

$$\Delta_{\mu_0}(p, m(0)) > \Delta_{\mu_0}(0, m) = 0,\ 0 \neq p \in \mathbb{T}^3$$

lar kelib chiqadi.

$\lim\limits_{z\to-\infty} \Delta_{\mu_0}(p, z) = 1$ va $\Delta_{\mu_0}(p, \cdot)$ funktsiya $(-\infty, u_{\min}(p))$ da monoton o'suvchi bo'lgani uchun, $\Delta_{\mu_0}(p, z)$ funktsiya $(m, u_{\min}(p))$ da yagona yechimga ega degan xulosaga kelamiz. Lemma 15.1 Teorema 14.14 ning (i) qismi isbotini yakunlaydi.

(ii) $\mu > \mu_0$ bo'lsin. Barcha $p \in \mathbb{T}^3,\ z \leq u_{\min}(p)$ uchun

$$\Delta_{\mu}(p, z) < \Delta_{\mu_0}(p, z)$$

tengsizlikka egamiz. Ixtiyoriy nolmas $p \in \mathbb{T}^3$ uchun $h_{\mu_0}(p)$ operator yagona qat'iy musbat $m < e_{\mu_0}(p)$ xos qiymatga ega va $0 < e_{\mu_0}(p) < u_{\min}(p),\ p \neq 0 \in \mathbb{T}^3$ tengsizlik o'rinli.

Lemma 15.1 dan $p \neq 0 \in \mathbb{T}^3$ larda $\Delta_{\mu_0}(p, e_{\mu_0}(p)) = 0$ ekanligi kelib chiqadi. $z < u_{\min}(p)$ uchun $\Delta_{\mu_0}(p, \cdot)$ funktsiya $(-\infty, u_{min}(p))$ da monoton kamayuvchi ekanligidan $\Delta_{\mu}(p, e_{\mu_0}(p)) < \Delta_{\mu_0}(p, e_{\mu_0}(p)) = 0, p \neq 0 \in \mathbb{T}^3$ va $\Delta_{\mu}(0, z) < \Delta_{\mu_0}(0, 0) = 0$ larga ega bo'lamiz. $\lim\limits_{z\to-\infty} \Delta_{\mu}(p, z) = 1$ tenglik va (16), Lemma 15.1 lar Teorema 14.14 isbotini yakunlaydi.

14.15 Teoremaning isboti $\Delta_{\mu}(p, z) = 1 - \Lambda(p, z) = 1 - D(p, \sqrt{u_{\min}(p) - w})$ va $z \leq m$ uchun $w = (m - z)^{1/2} \geq 0$ ekanligini hisobga olsak, Lemma 15.5 dan kelib chiqadi.

Natija 14.17 ning isboti Theorema 14.15 va 15.2, 15.3. Lemmalardan kelib chiqadi □.

Natija 14.19 ning isboti. $h_{\mu_0}(0)$ operator bo'sag'a energiya rezononsiga ega bo'lsin, u holda $\varphi(0) \neq 0$ bo'ladi(Lemma 15.2 ga qarang). O'z navbatida

(16.1) $$u_{\min}(p) = m + (l_1 l_2 - l^2)(2l)^{-1}(Up, p) + o(|p|^2) \quad \text{da} \quad p \to 0$$

asimptotika (Lemma 15.4 ning (ii) qismiga qarang) o'rinli va c_1, c_2 musbat sonlar uchun 14.17 natijadan (14.9) kelib chiqadi.

$\Delta_{\mu_0}(\cdot, m)$ funktsiyaning $\mathbb{T}^3 \setminus U_{\delta}(0)$ kompakt to'plamda musbatligi va uzluksizligi (14.10) ning o'rinli ekanligini ko'rsatadi. □

Natija 14.20 ning isboti. Lemma 15.3 ga ko'ra $\varphi(0) = 0$ va $\Delta_{\mu_0}(0, m) = 0$ ga ega bo'lamiz. $\mu_0 = \Lambda^{-1}(0, m)$ ekanligini hisobga olsak, (bunda $\Lambda(\cdot, \cdot)$ funktsiya (14.3) kabi aniqlangan)

$$\Delta_{\mu_0}(p, m) = \mu_0(\Lambda(0, m) - \Lambda(p, m))$$

tenglikka kelamiz. Va nihoyat, 14.12 talab isbotni yakunlaydi. □

17. ILOVA

Bu yerda biz Friedrichs modellari oilasining shunday bir muhim sinfi borligini (masalan, [6, 24] larga qarang) va bu sinf uchun 2 bo'limdagi barcha talablar bajarilishini ko'rsatamiz. $u(p, q)$ funktsiya (14.2) ko'rinishida bo'lishini talab qilamiz, ya'ni

$$u(p, q) = \varepsilon(p) + \varepsilon(p - q) + \varepsilon(q),$$

bunda $\varepsilon(p)$ funktsiya $\mathbb{T}^3$ da haqiqiy qiymatli va shartli manfiy aniqlangan. Shuning uchun bu funktsiya

(i) juft
(ii) $p = 0$ da minimumga ega.

Eslatib o'tamizki, agar kompleks qiymatli $\varepsilon : \mathbb{T}^3 \to \mathbb{C}$ chegaralangan funktsiya $\varepsilon(p) = \overline{\varepsilon(-p)}$ va ixtiyoriy $n \in \mathbb{N}$, barcha $p_1, p_2, .., p_n \in \mathbb{T}^3$ va $\sum_{i=1}^{n} z_i = 0$ ni qanoatlantiruvchi barcha $\mathbf{z} = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ lar uchun

(17.1) $$\sum_{i,j=1}^{n} \varepsilon(p_i - p_j) z_i \bar{z}_j \leq 0$$

shartni qanoatlantirsa, u shartli manfiy aniqlangan deyiladi.

Lemma 17.1. *F. q. $\varepsilon(\cdot)$ funktsiya to'r $\mathbb{T}^3$ da shartli manfiy aniqlangan, koordinatalar boshida yagona aynimagan minimumga ega va uning barcha 3-tartibli xususiy hosilalari uzluksiz hamda $\mathcal{B}(\theta, U_\delta(0))$ fazoga qarashli bo'lsin. U holda 14.2 va 14.12 talablar bajariladi.*

Proof. 14.12 talabning bajarilishini isbotlash yetarli.

(i) $\Lambda(p, u_{\min}(p))$ ning aniqlanishiga ko'ra,

$$\Lambda(p,u_{\min}(p)) - \Lambda(0,u_{\min}(0)) = \int\limits_{\mathbb{T}^3} \frac{2(u_0(t)-u_{\min}(0)) - [u_p(t)+u_{-p}(t)-2u_{\min}(p)]}{(u_p(t)-u_{\min}(p))(u_{-p}(t)-u_{\min}(p))(u_0(t)-u_{\min}(0)}\varphi^2(t)dt \tag{17.2}$$

ga ega bo'lamiz.

Faraz qilaylik $q_0(p)$ nuqta $u_p(\cdot)$ funktsiyaning minimum nuqtasi ya'ni $u_p(q_0(p)) = u_{min}(p)$ bo'lsin.

Yuqoridagi (17.2) tenglikda $t \to q+q_0(p)$ almashtirish olib, $q_0(-p) = -q_0(p)$ va $u_{-p}(q+q_0(-p)) = u_p(q+q_0(p))$ tengliklardan foydalanib, [6] da keltirilgan (5 Lemma) tengsizlikka ekvivalent bo'lgan va 14.12 talabning (i) qismini isbotlovchi

$$u_0(q) - u_{min}(0) > \frac{u_p(q_0(p)+q) + u_p(q_0(p)-q)}{2} - u_{min}(p)$$

tengsizlikka ega bo'lamiz.

(ii) Shartli manfiy aniqlangan ε funktsiya $s \neq 0$ bo'lganda $\hat{\varepsilon}(s)$ Fourier koeffitsiyentlarining nomusbatlik, ya'ni

$$\hat{\varepsilon}(s) \leq 0, \quad s \neq 0 \tag{17.3}$$

va $\sum_{s\in\mathbb{Z}^3\setminus\{0\}} \hat{\varepsilon}(s)$ qatorning absolyut yaqinlashuvchilik shartiga ekvivalent bo'lgan

$$\varepsilon(p) = \varepsilon(0) + \sum_{s\in\mathbb{Z}^3\setminus\{0\}} (cos(p,s)-1)\hat{\varepsilon}(s), \quad p \in \mathbb{T}^3, \tag{17.4}$$

(Lévy-Khinchin) yoyilmasiga keladi.

Qaralayotgan u va v funktsiyalar juft bo'lganligi uchun $\Lambda(\cdot)$ funktsiya ham juft. Shuning uchun

$$u_0(t) - \frac{u_p(t)+u_p(-t)}{2} = \sum_{s\in\mathbb{Z}^3\setminus\{0\}} \hat{\varepsilon}(s)(1+cos(t,s))(1-cos(p,s))$$

tengsizlikdan

$$\Lambda(0,m) - \Lambda(p,m) = \frac{1}{2}\sum_{s\in\mathbb{Z}^3\setminus\{0\}} (-\hat{\varepsilon}(s))(1-cos(p,s))\int\limits_{\mathbb{T}^3}(1+cos(t,s))F(p,t)dt + \tilde{B}(p), \tag{17.5}$$

yoyilma kelib chiqadi, bunda

$$F(p,\cdot) = \frac{[u_p(\cdot)+u_{-p}(\cdot)-2m]}{(u_p(\cdot)-m)(u_{-p}(\cdot)-m)(u_0(\cdot)-m)}v^2(\cdot)$$

va

$$\tilde{B}(p) = \frac{1}{4}\int\limits_{\mathbb{T}^3} \frac{[u_p(t)-u_{-p}(t)]^2}{(u_p(t)-m)(u_{-p}(t)-m)(u_0(t)-m)}v^2(t)dt.$$

$$B(p,s) = \int\limits_{\mathbb{T}^3}(1+\cos(t,s))F(p,t)dt$$

belgilash olamiz.

$B(p,s)$ funktsiyani quyidagi ikkita funktsiyaning yig'indisi sifatida yozib olamiz

$$B_\delta^{(1)}(p,s) = \int\limits_{\mathbb{T}^3\setminus U_\delta(0)} (1+\cos(t,s))F(p,t)dt$$

va

$$B_\delta^{(2)}(p,s) = \int\limits_{U_\delta(0)} (1+\cos(t,s))F(p,t)dt.$$

$\chi_\delta(\cdot)$ orqali $U_\delta(0)$ ning xarakteristik funktsiyasini belgilaymiz. $\delta > 0$ ni esa

$$mes\{(\mathbb{T}^3 \setminus U_\delta(0)) \cap supp\ v\} > 0$$

bo'ladigan shartdan tanlaymiz.

$F_\delta(p,\cdot) = (1-\chi_\delta(\cdot))F(p,\cdot)$ deb olamiz. U holda barcha $p \in \mathbb{T}^3$ va deyarli barcha

$$t \in (\mathbb{T}^3 \setminus U_\delta(0)) \cap supp\ v(\cdot)$$

lar uchun $F_\delta(\cdot,\cdot)$ funktsiya qat'iy musbat. u funktsiya $(0,0)$ da yagona minimumga ega va v funktsiya $p \in \mathbb{T}^3$ da uzluksiz bo'lganligidan $F_\delta(p,\cdot), p \in \mathbb{T}^3$ funktsiyaning Banach fazosiga tegishli ekanligiga kelamiz. U holda ba'zi (biror yetarlicha katta) $R > 0$ $c_1(\delta) > 0$ (yetarlicha kichik) va barcha $|s| \le R$, $p \in \mathbb{T}^3$ uchun

$$B_\delta^{(1)}(p,s) = \int\limits_{\mathbb{T}^3} (1+\cos(t,s))F_\delta(p,t)dt > c_1(\delta) > 0.$$

tengsizlikka ega bo'lamiz. Riemann-Lebeg lemmasiga ko'ra $s \to \infty$ da

$$B_\delta^{(1)}(p,s) = \int\limits_{\mathbb{T}^3} (1+\cos(t,s))F_\delta(p,t)dt \to \int\limits_{\mathbb{T}^3} F_\delta(p,t)dt > 0,\ p \in \mathbb{T}^3$$

ga kelamiz.

$$\tilde{F}(p) = \int\limits_{\mathbb{T}^3} F_\delta(p,t)dt$$

funktsiya $\mathbb{T}^3$ kompakt to'plamda uzluksizligidan barcha $p \in \mathbb{T}^3$ va $|s| > R$ lar uchun $B_\delta^{(1)}(p,s) \ge c_2(\delta)$ tengsizlik o'rinli ekanligi kelib chiqadi.

Shunday qilib, $c(\delta) = \min\{c_1(\delta), c_2(\delta)\}$ lar uchun $B_\delta^{(1)}(p,s) \ge c$ tengsizlik barcha $s \in \mathbb{Z}^3$, $p \in \mathbb{T}^3$ larda o'rinli. Shuningdek, $B_\delta^{(2)}(p,s) \ge 0$, $s \in \mathbb{Z}^3$, $p \in \mathbb{T}^3$ dan $B(p,s) > c$, $s \in \mathbb{Z}^3$, $p \in \mathbb{T}^3$ ekanligi kelib chiqadi. $\tilde{B}(p) \ge 0$, $p \in \mathbb{T}^3$ va $\hat{\varepsilon}(s) \le 0$, $s \in \mathbb{Z}^3 \setminus \{0\}$ tengsizlikni hisobga olgan holda ((17.3) ga qarang) (17.4) va (17.5) yoyilmalardan

$$\Lambda(0,m) - \Lambda(p,m) \ge c(\varepsilon(p) - \varepsilon(0))$$

ga ega bo'lamiz. Bular $\varepsilon(\cdot)$ haqidagi farazlar bilan birgalikda Lemma 17.1 ning isbotini yakunlaydi. $\square$

ADABIYOTLAR

REFERENCES

[1] S. Albeverio, S.N. Lakaev, Abdullaev J.I. Estimates on the number of eigenvalues of two-particle discrete Schrodinger operators. arXiv:math-ph/0501036 v1 12 Jan 2005.

[2] S. Albeverio, S.N. Lakaev, Abdullaev J.I. On the spectral properties of two-particle discrete Schrödinger operators. Preprint. Bonn SFB 611, 2004.N 170. pp.1-14.

[3] Albeverio, S., Gesztesy F., and Høegh-Krohn R.: The low energy expansion in non-relativistic scattering theory. Ann. Inst. H. Poincaré Sect. A (N.S.) **37**, 1–28 (1982).

[4] Albeverio S., Høegh-Krohn R., and Wu T. T.: A class of exactly solvable three-body quantum mechanical problems and universal low energy behavior. Phys. Lett. A **83**, 105–109 (1971).

[5] Albeverio S., Gesztesy F., Høegh-Krohn R., and Holden H.: *Solvable Models in Quantum Mechanics*. Springer-Verlag, New York, 1988; 2nd ed. (with an appendix by P. Exner), Chehea, AMS, 2004.

[6] S. Albeverio, S. N. Lakaev,K. A. Makarov, Z. I. Muminov: The Threshold Effects for the Two-particle Hamiltonians on Lattices, Comm.Math.Phys. **262**(2006), 91–115 .

[7] S. Albeverio, S. N. Lakaev, K. A. Makarov: The Efimov Effect and an Extended Szegö-Kac Limit Theorem, Letters in Math. Phys, V. **43** (1998), 73-85.

[8] S. Albeverio, S. N. Lakaev and Z. I. Muminov: Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics. Ann. Henri Poincaré. **5**, (2004),743–772.

[9] S. Albeverio, S.N. Lakaev, Z.I. Muminov, The threshold effects for a family of Friedrichs models under rank one perturbations. J. Math. Anal. **330**, (2007),1152–1168

[10] S. Albeverio, S.N. Lakaev, Z.I. Muminov: On the number of eigenvalues of a model operator associated to a system of three-particles on lattices. (submitted to Revista Matemetica Iberoamericana, 2006)

[11] S. Albeverio, S.N. Lakaev, Z.I. Muminov: On the spectrum of a model operator associated to a system of three-particles on lattices. arXiv:math-ph/0508029 v1 14 Aug 2005 8.

[12] S. Albeverio, S.N. Lakaev, Z.I. Muminov: On the number of eigenvalues of a model operator associated to a system of three-particles on lattices, arXiv: math- ph/ 0508029 v2 14 Aug 2006

[13] S. Albeverio, S.N. Lakaev, K.A.Makarov and Z.I. Muminov: The Threshold effects for the two-particle Hamiltonians on lattices. arXiv:math-ph/0501013 v1 6 Jan 2005.

[14] S. Albeverio, S.N. Lakaev, Z.I. Muminov: On the spectrum of a model operator associated to a system of three-particles on lattices. arXiv:math-ph/0508029 v1 14 Avg 2005 4.

[15] S. Albeverio, S.N. Lakaev, R.Kh. Djumanova : On the essential and discrete spectrum of a model operator related to three-particle discrete Schrodinger operators. arXiv:math-ph/0501024 v1 11 Jan 2005.

[16] S. Albeverio, S.N. Lakaev, R.Kh. Djumanova On the essential and discrete spectrum of a model operator. Preprint. Bonn SFB 611, 2004.N 170. pp.1-25.

[17] S. Albeverio, S. N. Lakaev and Z. I. Muminov: On the structure of the essential spectrum for the three particle Schrödinger operators on lattices. Math.Nachr.**280**(2007), No7, 1–18

[18] Albeverio S. , Lakaev S. N. and Muminov Z. I. : The threshold effects for a family of Friedrichs models under rank one perturbations. arXiv:math.SP/0604277 v1 12 Apr 2006

[19] Albeverio S. , Lakaev S. N. and Muminov Z. I. : On the number of eigenvalues of a model operator associated to a system of three-particles on lattices. arXiv:math-ph/0508029 v2 14 Aug 2006

[20] S. Albeverio, S. N. Lakaev, K.A.Makarov and Z.I.Muminov: Low-energy effects for the two-particle operators on a lattice. Preprint. Bonn SFB 611, 2004.N 126. pp.1-16.

[21] Sergio Albeverio, Saidakhmad N. Lakaev, Tulkin H. Rasulov: The Efimov Effect for a Model Operator Associated to a System of three non Conserved Number of Particles, Submitted to " Methods of Functional Analysis and Topology" ., Accepted 10.01.07

[22] R. D. Amado and J. V. Noble: Efimov effect; A new pathology of three-particle Systems, II. Phys. Lett. **B.35.** No.1, 25-27, (1971); Phys. Lett. **D.5.** No.8, (1972), 1992-2002.

[23] Berg C., Christensen J. P. R, and Ressel P.: *Harmonic analysis on semigroups. Theory of positive definite and related functions.* Graduate Texts in Mathematics, Springer-Verlag, New York, 1984. 289 pp.

[24] Carmona R. and Lacroix J.: *Spectral theory of random Schrödinger operators. Probability and its Applications*, 1990, Birkhäuser Boston.

[25] G. F. Dell'Antonio, R. Figari, A. Teta: Hamiltonians for systems of N particles interacting through point interactions, Ann. Inst. H. Poincaré Phys. Théor. **60** (1994), no. 3, 253–290.

[26] Jensen A. and Kato T.: Spectral properties of Schrödinger operators and time-decay of the wave functions. Duke Math. J. **46**. 583–611 (1979).

[27] Graf G. M. and Schenker D.: 2-magnon scattering in the Heisenberg model. Ann. Inst. H. Poincaré Phys. Théor. **67**, 91–107 (1997).

[28] Klaus M. and Simon B.: Coupling constants thresholds in non-relativistic quantum mechanics. I. Short range two body case. Ann. Phys. **130**, 251–281 (1980).

[29] Kostrykin V. and Schrader R.: Cluster properties of one particle Scrödinger operators. II. Rev. Math. Phys. **10**, 627–682 (1998).
[30] Faddeev L. D. : On a model of Friedrichs in the theory of perturbations of the continuous spectrum(Russian). Trudy Mat. Inst. Steklov **73**(1964), 292–313.
[31] L. D. Faddeev and S. P. Merkuriev: Quantum scattering theory for several particle systems, Kluwer Academic Publishers, 1993.
[32] M. V. Fedoryuk: Asymptotics of integrals and series [in Russian], Nauka, Moscow (1987).
[33] P. A. Faria da Veiga, L. Ioriatti and M. O'Carroll: Energy-momentum spectrum of some two-particle lattice Schrödinger Hamiltonians, Phys. Rev. E (3) **66**, (2002), 016130, 9 pp.
[34] Friedrichs K. O.: On the perturbation of continuous spectra. Communications on Appl. Math. **1**(1948), 361–406.
[35] V. Efimov: Energy levels of three resonantly interacting particles, Nucl. Phys. A **210** (1973), 157–158.
[36] Yu. G. Kondratiev and R. A. Minlos: One-particle subspaces in the stochastic XY model, J. Statist. Phys. **87** (1997), 613–642.
[37] Lakaev S. N. : Some spectral properties of the generalized Friedrichs model, (Russian) Trudy Sem. Petrovsk. No. 11 (1986), 210–238, 246, 248; translation in J. Soviet Math. **45** (1989), no. 6, 1540–1565.
[38] S. N. Lakaev: On an infinite number of three-particle bound states of a system of quantum lattice particles, Theor. and Math. Phys.**89** (1991), No.1, 1079–1086.
[39] S. N. Lakaev: Bound states and resonances fo the N-particle discrete Schrödinger operator, Theor. Math. Phys. **91** (1992), No.1, 362-372.
[40] S. N. Lakaev: The Efimov's Effect of a system of Three Identical Quantum lattice Particles, Funkcionalnii analiz i ego priloj. , **27** (1993), No.3, pp.15-28, translation in Funct. Anal.Appl.
[41] S. N. Lakaev and J. I. Abdullaev: The spectral properties of the three-particle difference Schrödinger operator, Funct.Anal. Appl. **33** (1999), No. 2, 84-88.
[42] S.N.Lakaev and M.Kh.Shermatov:On infiniteness of the discrete spectrum of Hamiltonian of system of three-particles(two fermion and other), Uspexi matem.nauk **54**(1999), 165-166.
[43] Lakaev S. N.: Discrete spectrum and resonances of the one-dimensional Schrödinger operator for small coupling constants. Teoret. Mat. Fiz. **44**, 381–386 (1980).
[44] Lakshtanov E. L. and Minlos R. A. : The spectrum of two-particle bound states of transfer matrices of Gibbs fields (an isolated bound state). (Russian) Funktsional. Anal. i Prilozhen. **38**(2004), No.3, 52–69; (translation in Funct. Anal. Appl. 38 (2004), No. 3, 202–216)
[45] Lakshtanov E. L. and Minlos R. A. : The spectrum of two-particle bound states of transfer matrices of Gibbs fields (fields on a two-dimensional lattice: adjacent levels). (Russian) Funktsional. Anal. i Prilozhen. **39** (2005), No. 1, 39–55, (translation in Funct. Anal. Appl. 39 (2005), no. 1, 31–45)
[46] V. A. Malishev and R. A. Minlos: Linear infinite-particle operators. Translations of Mathematical Monographs, 143. AMS, Providence, RI.
[47] D. C. Mattis: The few-body problem on lattice, Rev.Modern Phys. **58** (1986), No. 2, 361-379.
[48] Mogilner A.: Hamiltonians in solid state physics as multi-particle discrete Schrödinger operators: Problems and results. Advances in Soviet Mathematics **5**, 139–194 (1991).
[49] R. A. Minlos and Y. M. Suhov: On the spectrum of the generator of an infinite system of interacting diffusions, Comm. Math. Phys. **206** (1999), 463–489.
[50] Ovchinnikov Yu. N. and Sigal I. M.: Number of bound states of three-particle systems and Efimov's effect. Ann. Physics **123**, 274–295 (1989).
[51] Rauch J.: Perturbation theory for eigenvalues and resonances of Schrödinger Hamiltonians. J. Funct. Anal. **35**, 304–315 (1980).
[52] M. Reed and B. Simon: Methods of modern mathematical physics. III: Scattering teory, Academic Press, N.Y., 1979.
[53] M. Reed and B. Simon: Methods of modern mathematical physics. IV: Analysis of Operators, Academic Press, N.Y., 1979.
[54] Simon B.: Large time behavior of the L^p norm of Schrödinger Semigroups. J. Funct. Anal. **40**, 66–83 (1981).
[55] A. V. Sobolev: The Efimov effect. Discrete spectrum asymptotics, Commun. Math. Phys. **156** (1993), 127–168.
[56] H. Tamura: The Efimov effect of three-body Schrödinger operator, J. Funct. Anal. **95** (1991), 433–459.
[57] H. Tamura: Asymptotics for the number of negative eigenvalues of three-body Schrödinger operators with Efimov effect. Spectral and scattering theory and applications, Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo
[58] Tamura H.: The Efimov effect of three-body Schrödinger operators: Asymptotics for the number of negative eigenvalues. Nagoya Math. J. **130**, 55–83(1993).
[59] Yafaev D. R.: *Scattering theory: Some old and new problems*, Lecture Notes in Mathematics, 1735. Springer-Verlag, Berlin, 2000, 169 pp.
[60] D. R. Yafaev: On the theory of the discrete spectrum of the three-particle Schrödinger operator, Math. USSR-Sb. **23** (1974), 535–559.
[61] Yafaev D. R.: The virtual level of the Schrödinger equation. J. Soviet. Math., **11**, 501–510 (1979).
[62] Zhizhina E. A.: Two-particle spectrum of the generator for stochastic model of planar rotators at high temperatures. J. Statist. Phys. **91**, 343–368 (1998).
[63] X. P . Wang: On the existence of the N- body Efimov effect, J. Funct. Anal. **95** (2001), 433–459.